Lecture 5

Last time: In spherical coordinates, $u(x) = U(r,\theta)$,

$$\Delta u(r,\theta) = U_{rr} + \frac{n-1}{r}U_r + \frac{1}{r}^2\Delta_{S^{n-1}}U.$$

If $U(r,\theta) = f(r)B(\theta)$, then

$$\Delta u = (f_{rr} + \frac{n-1}{r}f_r)B(\theta) + \frac{1}{r^2}f\Delta_{S^{n-1}}B.$$

Proposition 1 *Eigenvalues of $\Delta_{S^{n-1}}$ are $-k(k+n-2)$, where $k \geq 0$, so $\lambda_1(S^{n-1}) = n-1$.*

Let $\Delta_{S^{n-1}}B_k(\theta) = -k(k+n-2)B_k(\theta)$, then

$$\Delta(f(r)B_k(\theta)) = (f_{rr} + \frac{n-1}{r}f_r - k(k+n-2)\frac{f_r}{r^2})B(\theta).$$

From this we get that harmonic functions which has form r^pB_k must satisfies $p = k$ or $-k-n+2$, thus we get a gap for the values of p:

$$\cdots\cdots, -n, -(n-1), -(n-2), \blacksquare, 0, 1, 2, \cdots\cdots$$

The terms before the gap are harmonic functions blowing faster than or as fast as the Green's function, and the terms after the gap correspond to homogenous harmonic polynomials. The gap comes from the removable singularity theorem.

Laplacian in inverted coordinates: Kelvin transform I

First we define the inverted transform $T : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$ to be $Ty = \frac{y}{|y|^2} = x$. Note that $T = T^{-1}$. Let's find the component of the metric tensor of y.

Now the metric of coordinates x^i is $g_{Euc} = (\delta_{ij})$. Let $e_k = \frac{\partial}{\partial y^k}$. Suppose $T_*e_k = \sum a_{ik}\frac{\partial}{\partial x^i}$, then $a_{ik} = \frac{\partial x^i}{\partial y^k}$, and

$$\begin{aligned}(T^*g_{Euc})_{kl} = T^*\delta_{ij}(e_k, e_l) &= \delta_{ij}(T_*e_k, T_*e_l)\\ &= \delta_{ij}(\sum a_{mk}\frac{\partial}{\partial x^m}, \sum a_{rl}\frac{\partial}{\partial x^r})\\ &= \sum a_{mk}a_{ml}\end{aligned}$$

Because

$$a_{mk} = \frac{\partial x^m}{\partial y^k} = \frac{\partial}{\partial y^k}(\frac{y^m}{|y|^2}) = \frac{\delta_{km}}{|y|^2} - \frac{2y^ky^m}{|y|^4},$$

so we have

$$\begin{aligned} g_{kl} = (T^* g_{Euc})_{kl} &= \sum_m a_{mk} a_{ml} \\ &= \sum_m (\frac{\delta_{km}}{|y|^2} - \frac{2y^k y^m}{|y|^4})(\frac{\delta_{lm}}{|y|^2} - \frac{2y^l y^m}{|y|^4}) \\ &= \frac{\delta_{kl}}{|y|^4} - \frac{2y^l y^k + 2y^l y^k}{|y|^6} + \frac{4y^l y^k |y|^2}{|y|^8} \\ &= \frac{\delta_{kl}}{|y|^4}. \end{aligned}$$

Thus

$$det(g_{kl}) = \frac{1}{|y|^4 n} = |y|^{-4n}, \qquad g^{kl} = |y|^4 \delta_{kl}.$$

So by the formula in last lecture, we get

$$\begin{aligned} \Delta u(Ty) &= |y|^{2n} \partial_i (\delta_{ij} |y|^4 |y|^{-2n} \partial_j u) \\ &= |y|^{2n} \partial_i (|y|^{4-2n} \partial_i u) \\ &= |y|^{2n} |y|^{4-2n} \partial_i \partial_i u + 2(2-n)|y|^2 < \nabla u, y > \\ &= |y|^4 u_{ii} + 2(2-n)|y|^2 u_i y_i. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |y|^{n+2} \sum \frac{\partial^2}{\partial {Y^i}^2} (|y|^{2-n} u) &= |y|^{n+2} ((\sum \frac{\partial^2}{\partial {y^i}^2} |y|^{2-n}) u + 2\nabla |y|^{2-n} \cdot \nabla u + |y|^{2-n} u_{ii}) \\ &= 2(2-n)|y|^2 u_i y^i + u_{ii} |y|^4, \end{aligned}$$

so we get

$$\mathbf{\Delta u(Ty) = |y|^{n+2} \Delta (|y|^{2-n} u)}.$$

Definition 1 *The* **Kelvin Transform** *of u is defined to be $Ku(y) = |y|^{2-n} u(\frac{y}{|y|^2})$.*

Corollary 1 *If u is harmonic on $\mathbb{R}^n - \{0\}$, then Ku is harmonic on $\mathbb{R}^n - \{0\}$.*

Now let's look at the degree gap again:

$$\cdots\cdots, -n, -(n-1), -(n-2), \blacksquare, 0, 1, 2, \cdots\cdots$$

As we have known, those terms after "$\blacksquare$" correspond to homogeneous harmonic polynomials $u_k = r^k B_k(\theta)$. Apply the Kelvin's transform to this harmonic functions, we get

$$Ku_k(y) = |y|^{2-n} u_k(\frac{y}{|y|^2}) = r^{2-n} r^{-k} B_k(\theta) = r^{2-n-k} B_k(\theta).$$

Thus those terms before "$\blacksquare$", i.e. those blow up faster than the Green's near origin, are just the Kelvin transform of the homogeneous harmonic polynomials.

Also we can know that if we apply Kelvin's transform to homogeneous harmonic polynomials, then near ∞, we get harmonic functions which decays like $\frac{1}{r^{n-2}}, \frac{1}{r^{n-1}}, \frac{1}{r^n}$, and there is no harmonic function which decay like $\frac{1}{r^{n-3}}$ near ∞.

Harmonic at ∞

Definition 2 *Suppose u is harmonic on $\mathbb{R}^n \setminus K$, where K is a compact set, then we say u is* **harmonic at ∞** *if its Kelvin transform Ku is harmonic at the origin.*

Remark 1 *1) We have to remove a compact set K, otherwise if u is harmonic on $\mathbb{R}^n$ and harmonic at ∞, then u is bounded in $\mathbb{R}^n$, thus u is constant, which we have no interest.*
2) u is harmonic at $\infty \Longrightarrow Ku$ is harmonic at origin $\Longrightarrow Ku(x) = |x|^{2-n}u(\frac{x}{|x|^2}) \leq C \Longrightarrow u(y) \leq \frac{C}{|y|^{n-2}}$, i.e. u decay at least as fast as the Green's function Γ.

Theorem 1 *Suppose $n > 2, u$ is harmonic in $\mathbb{R}^n \setminus K$, where K is compact subset. If $\lim_{x\to\infty} u(x) = 0$, then u is harmonic at ∞.*

Proof: $Ku(x) = |x|^{2-n}u(\frac{x}{|x|^2}) = o(|x|^{2-n})$ as $x \to 0$, thus Ku has harmonic extension to 0 by Removable Singularity Theorem. ■

Corollary 2 *If u harmonic on $\mathbb{R}^n \setminus K$, and $\lim_{x\to\infty} u = 0$, then $u(x) \leq \frac{C}{|x|^{n-2}}$.*

This corollary tells us that harmonic function which decays at ∞, must decay at least as fast as the Green's function: another "gap".

Now we turn to the "$n = 2$" case.

Theorem 2 *Suppose $n = 2, u$ is harmonic in $\mathbb{R}^n \setminus K$, where K is compact subset. If $\lim_{x\to\infty} \frac{u(x)}{\log|x|} = 0$, then u is harmonic at ∞.*

Proof: Just the same as last theorem. ■

Corollary 3 *$n = 2$. If u harmonic on $\mathbb{R}^n \setminus K$, and $\lim_{x\to\infty} \frac{u(x)}{\log|x|} = 0$, then u has a limit at ∞.*

Kelvin II: Poission integral formula proof.

Poission Integral Formula: If u is harmonic on $\mathbb{R}^n$, then

$$u(x) = \int_{\partial B} P(x,y)u(y)d\sigma_y,$$

where $P(x,y) = \frac{1-|x|^2}{|x-y|^n}\frac{1}{n\omega_n}$. So

$$Ku(x) = |x|^{2-n}u(\frac{x}{|x|^2}) = \int_{\partial B}|x|^{2-n}P(\frac{x}{|x|^2},y)u(y)d\sigma_y.$$

Claim: $|x|^{2-n}P(\frac{x}{|x|^2},y) = -P(x,y)$.

In fact, by using the formula $|\frac{y}{|y|}-|y|x| = |\frac{x}{|x|}-|x|y|$, we get

$$\begin{aligned}
|x|^{2-n}P(\frac{x}{|x|^2},y) &= |x|^{2-n}\frac{1-\frac{1}{|x|^2}}{|\frac{x}{|x|^2}-y|^n}\frac{1}{n\omega_n} \\
&= |x|^{2-n}\frac{1-\frac{1}{|x|^2}}{\frac{1}{|x|^n}|\frac{y}{|y|}-|y|x|^n}\frac{1}{n\omega_n} \\
&= \frac{|x|^2-1}{|\frac{y}{|y|}-|y|x|^n}\frac{1}{n\omega_n}.
\end{aligned}$$

But $|y|=1$, so we get

$$|x|^{2-n}P(\frac{x}{|x|^2},y) = -P(x,y).$$

By this claim, we have

$$Ku(x) = -\int_{\partial B}P(x,y)u(y)d\sigma_y, \qquad \forall |x|>1.$$

So $Ku(x)$ is harmonic on $|x|>1$.

But Ku is analytic, thus ΔKu is also analytic, so $\Delta Ku \equiv 0$ on $\mathbb{R}^n - \{0\}$, which means that Ku is harmonic.