

Lecture 8

A Priori Estimates for Poisson's Equation.

Recall that $Nf = \int_{\Omega} \Gamma(x-y)f(y)dy$ is called the Newtonian Potential of f .

Proposition 1 *Suppose Ω is bounded domain, $f \in L^1(\Omega)$, and $\omega = Nf$ is the Newtonian potential of f . Then $\omega \in C^1(\mathbb{R}^n)$ and*

$$D_i\omega(x) = \int_{\Omega} D_i\Gamma(x-y)f(y)dy, \forall x \in \Omega.$$

Proof: $\Gamma = C|x|^{2-n} \implies |D\Gamma| \leq C|x|^{1-n}$, therefore

$$v(x) = \int_{\Omega} D_i\Gamma(x-y)f(y)dy$$

is well defined. ($|v(x)| \leq \|f\|_{L^\infty} \int_{\Omega} |D_i\Gamma|dy \leq C\|f\|_{L^\infty}$.)

Define $\eta_\epsilon(t)$ to be C^∞ function with properties: (1) $\eta_\epsilon(t) = 0$ for $t < \epsilon$; (2) $\eta_\epsilon(t) = 1$ for $t > 2\epsilon$; (3) $0 \leq \eta_\epsilon(t) \leq 1$; (4) $|D\eta_\epsilon| \leq \frac{2}{\epsilon}$. Define $\omega_\epsilon(x)$ to be

$$\omega_\epsilon(x) = \int_{\Omega} \Gamma(x-y)\eta_\epsilon(|x-y|)f(y)dy.$$

Then $\omega_\epsilon(x) \in C^1$ and $\omega_\epsilon(x) \rightarrow \omega$ uniformly.

$$\begin{aligned} v(x) - D_i\omega_\epsilon(x) &= \int_{\Omega} (D_i\Gamma(x-y) - D_i(\Gamma(x-y)\eta_\epsilon(|x-y|)))f(y)dy \\ &= \int_{\Omega} D_i((1 - \eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy \\ &= \int_{|x-y| \leq 2\epsilon} D_i((1 - \eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy \end{aligned}$$

So

$$\begin{aligned} |v(x) - D_i\omega_\epsilon(x)| &\leq \sup |f| \int_{|x-y| \leq 2\epsilon} \left(\frac{2}{\epsilon} |\Gamma(x-y)| + |D_i\Gamma(x-y)| \right) dy \\ &\leq \sup |f| \left(\frac{2}{\epsilon} \int_{|z| \leq 2\epsilon} |\Gamma(z)| dz + \int_{|z| \leq 2\epsilon} |D_i\Gamma(z)| dz \right) \\ &\leq \sup |f| \left(\frac{2}{\epsilon} \int_{|z| \leq 2\epsilon} \frac{C}{|z|^{n-2}} dz + \int_{|z| \leq 2\epsilon} \frac{C}{|z|^{n-1}} dz \right) \\ &\leq C \sup |f| \left(\frac{2}{\epsilon} \int_{r \leq \epsilon} r dr + \int_{r \leq \epsilon} dr \right) \\ &\leq C \cdot \epsilon \sup |f|. \end{aligned}$$

Now we have $\omega_\epsilon \rightarrow \omega$ and $D_i\omega_\epsilon \rightarrow v$ uniformly on compact subsets as $\epsilon \rightarrow 0$, thus $\omega \in C^1(\mathbb{R}^n)$ and $D_i\omega = v$. ■

Theorem 1 Let $u \in C^2(\overline{\Omega})$, $f \in L^\infty(\Omega)$ and $\Delta u = f$ in Ω . Then for any compact subdomain $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^1(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}).$$

Proof: Let ω be the Newtonian potential of f , i.e. $\omega(y) = \int_{\Omega} \Gamma(x-y)f(x)dx$. Then from Green's representation formula,

$$v(y) = u(y) - \omega(y) = \int_{\partial\Omega} u(x) \frac{\partial}{\partial\nu_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial\nu} d\sigma_x$$

is a harmonic function. So

$$\|\omega\|_{C^0(\Omega')} = \sup_{y \in \Omega'} \left| \int_{\Omega} \Gamma(x-y)f(x)dx \right| \leq \|f\|_{L^\infty(\Omega)} \sup_{y \in \Omega'} \int_{\Omega} \frac{C}{|x-y|^{n-2}} ds \leq C\|f\|_{L^\infty(\Omega)},$$

and

$$\begin{aligned} D_i \omega(y) &= \int_{\Omega} D_i \Gamma(y-x)f(x)ds \\ &\leq \|f\|_{L^\infty(\Omega)} \int_{\Omega} D_i \Gamma(y-x)ds \\ &\leq \|f\|_{L^\infty(\Omega)} \frac{C}{|x-y|^{n-1}} ds \\ &= C\|f\|_{L^\infty(\Omega)} \end{aligned}$$

Thus $\|D_i \omega\|_{C^0(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$, and so

$$\|\omega\|_{C^1(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}.$$

Since v is harmonic, we have

$$\begin{aligned} \|Dv\|_{C^0(\Omega')} &\leq C\|v\|_{C^0(\Omega)} \\ &\leq C(\|u\|_{C^0(\Omega)} + \|\omega\|_{C^0(\Omega)}) \\ &\leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}). \end{aligned}$$

Thus

$$\begin{aligned} \|u\|_{C^1(\Omega')} &\leq \|v\|_{C^1(\Omega')} + \|\omega\|_{C^1(\Omega')} \\ &\leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}). \end{aligned} \quad \blacksquare$$

More over, one can show that for any $0 \leq \alpha < 1$,

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(\|u\|_{C^0} + \|f\|_{L^\infty}).$$

This is not true for $\alpha = 1$.

But if $f \in C^\alpha(\bar{\Omega})$, then

$$\|u\|_{C^2(\Omega')} \leq C(\|u\|_C^0 + \|f\|_{L^\infty(\Omega)})$$

and

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_C^\alpha + \|f\|_{L^\infty(\Omega)}).$$

$C^{1,\alpha}$ estimate for Newtonian Potential (Ω Bounded)

$$\omega(x) = \int_{\Omega} \Gamma(x-y)f(y)dy \implies D_i\omega(x) = \int_{\Omega} D_i\Gamma(x-y)f(y)dy.$$

Theorem 2 *If $f \in L^\infty$, then $\omega \in C^{1,\alpha}(\Omega)$.*

Proof: Take $x, \bar{x} \in \Omega$, let $\delta = |x - \bar{x}|$, and $\xi = \frac{1}{2}(x + \bar{x})$.

$$\begin{aligned} D_i\omega(x) - D_i\omega(\bar{x}) &= \int_{\Omega} (D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y))f(y)dy \\ &\leq \|f\|_{L^\infty(\Omega)} \int_{\Omega} |D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)|dy \\ &\leq \|f\|_{L^\infty(\Omega)} \left(\int_{B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)|dy \right. \\ &\quad \left. + \int_{\Omega - B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)|dy \right) \\ &= \|f\|_{L^\infty(\Omega)}(I + II), \end{aligned}$$

where

$$\begin{aligned} I &\leq \int_{B_\delta(\xi)} |D_i\Gamma(x-y)|dy + \int_{B_\delta(\xi)} |D_i\Gamma(\bar{x}-y)|dy \\ &\leq \int_{B_{\frac{3\delta}{2}}(x)} |D_i\Gamma(x-y)|dy + \int_{B_{\frac{3\delta}{2}}(\bar{x})} |D_i\Gamma(\bar{x}-y)|dy \\ &\leq C \int_{B_{\frac{3\delta}{2}}(x)} \frac{1}{|x-y|^{n-1}}dy + \int_{B_{\frac{3\delta}{2}}(\bar{x})} \frac{1}{|\bar{x}-y|^{n-1}}dy \\ &\leq C \cdot \frac{3\delta}{2} \\ &= C|x - \bar{x}|, \end{aligned}$$

and

$$\begin{aligned} II &= \int_{\Omega - B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)|dy \\ &\leq |x - \bar{x}| \int_{\Omega - B_\delta(\xi)} |DD_i\Gamma(\hat{x}-y)|dy \\ &\leq C \cdot \delta \int_{|y-\xi| \geq \delta} \frac{1}{|\hat{x}-y|^n}dy \end{aligned}$$

Since we have

$$\begin{aligned}
|y - \xi| &\leq |y - \hat{x}| + |\hat{x} - \xi| \\
&\leq |y - \hat{x}| + \frac{\delta}{2} \\
&\leq |y - \hat{x}| + \frac{1}{2}|y - \xi| \\
\implies \frac{1}{2}|y - \xi| &\leq |y - \hat{x}|.
\end{aligned}$$

Thus

$$\begin{aligned}
II &\leq C \cdot \delta \int_{|y-\xi| \geq \delta} \frac{1}{|y-\xi|^n} dy \\
&\leq C \cdot \delta \int_{\delta}^R \frac{1}{r} dr \\
&\leq C \cdot \delta (\log R - \log \delta) \\
&\leq C \cdot \delta (\log R + c\delta^{\alpha-1}) \quad (\because -\log \delta \leq C\delta^{\alpha-1} \text{ for } 0 \leq \alpha < 1) \\
&= C\delta + C\delta^{\alpha} \leq C\delta^{\alpha} \\
&= C|x - \bar{x}|^{\alpha}.
\end{aligned}$$

Combine the above results, we get $\omega \in C^{1,\alpha}(\Omega)$.

Further more, we get the $C^{1,\alpha}$ estimate

$$\|\omega\|_{C^{1,\alpha}(\Omega)} \leq C\|f\|_{L^{\infty}(\Omega)}, \quad 0 \leq \alpha < 1. \quad \blacksquare$$

Just as last time, this implies

Corollary 1 *Suppose $\Delta u = f, f \in L^{\infty}(\Omega), \Omega' \subset\subset \Omega$, then for $0 \leq \alpha < 1$,*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^{\infty}(\Omega)}).$$

Remark 1 *Look at the above proof and assume $f \in L^p(\Omega)$.*

$$\begin{aligned}
D_i \omega(x) - D_i \omega(\bar{x}) &= \int_{\Omega} (D_i \Gamma(x-y) - D_i \Gamma(\bar{x}-y)) f(y) dy \\
&\leq \left\{ \int_{\Omega} |D_i \Gamma(x-y) - D_i \Gamma(\bar{x}-y)|^q dy \right\}^{1/q} \left\{ \int_{\Omega} |f(y)|^p dy \right\}^{1/p} \\
\frac{1}{p} + \frac{1}{q} = 1 &\implies q = \frac{p}{p-1}.
\end{aligned}$$

In the 2^{nd} part of the above proof, we had

$$\begin{aligned}
II &\leq \left\{ \int_{\Omega} \|x - \hat{x}\| |DD_i \Gamma(\hat{x}-y)|^q \right\}^{1/q} \\
&\leq C \cdot \delta \left(\int_{|y-\xi| \geq \delta} \frac{1}{|y-\xi|^{nq}} \right)^{1/q} \\
&\leq C \cdot \delta \cdot (\delta^{-nq+n})^{1/q} \\
&= C \cdot \delta^{1-n+\frac{n}{q}}.
\end{aligned}$$

Let $\alpha = 1 - n + n\frac{p-1}{p}$, then $p(\alpha - 1 + n) = pn - n$, i.e. $p = \frac{n}{1-\alpha}$, we have:

If $\Delta u = f, f \in L^p(\Omega), p = \frac{n}{1-\alpha}, \Omega' \subset\subset \Omega$, then

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{C^0(\Omega)}).$$

Later we will show

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{C^0(\Omega)}).$$

So $C^{1,\alpha}$ estimate follows by Sobolev Embedding theorem.