Lecture 9

C^2 estimates

Proposition 1 *Let f be bounded and $f \in C^\alpha(\Omega)$ (locally), $\omega(x) = \int_\Omega \Gamma(x-y)f(y)dy$ be the Newtonian potential of f. Then $\omega \in C^2(\Omega)$, $\Delta\omega = f$ and*

$$D_{ij}\omega(x) = \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j d\sigma_y,$$

where Ω_0 is nice domain (for instance, has C^2 boundary) with $\Omega_0 \supset \Omega$, and we just extend f to vanish outside of Ω.

Proof: Define $u(x) = \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j d\sigma_y$, the right hand side of above. Since $D_{ij}\Gamma(x-y) \leq \frac{1}{|x-y|^n}$, $|f(y)-f(x)| \leq \|f\|_{C^\alpha}|x-y|^\alpha$, $u(x)$ is well defined.

Take $\eta_\varepsilon(x) \in C^\infty(\mathbb{R})$ such that (1) $0 \leq \eta_\varepsilon(x) \leq 1$; (2) $\eta_\varepsilon(x) = 0$ for $x \leq \varepsilon$; (3) $\eta_\varepsilon(x) = 1$ for $x \geq 2\varepsilon$; (4) $|\nabla\eta_\varepsilon| \leq \frac{2}{\varepsilon}$. Let

$$v_\varepsilon(x) = \int_\Omega D_i\Gamma(x-y)\eta_\varepsilon(|x-y|)f(y)dy.$$

Claim: $v_\varepsilon \in C^1(\Omega)$, and $v_\varepsilon \to D_i\omega$ uniformly in Ω.

In fact, we have

$$\begin{aligned}
D_j v_\varepsilon(x) &= \int_\Omega D_j(D_i\Gamma(x-y)\eta_\varepsilon(|x-y|))f(y)dy \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\varepsilon(|x-y|))(f(y)-f(x))dy \\
&\qquad + f(x)\int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\varepsilon(|x-y|))dy \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\varepsilon(|x-y|))(f(y)-f(x))dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i(\Gamma(x-y)\eta_\varepsilon(|x-y|))\nu_j(y)ds_y \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\varepsilon(|x-y|))(f(y)-f(x))dy - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)ds_y.
\end{aligned}$$

Now

$$
\begin{aligned}
|u(x) - D_j v_\varepsilon(x)| &\le \int_{|x-y|\le 2\varepsilon} D_j\{(1-\eta_\varepsilon)D_i\Gamma\}(f(y)-f(x))dy \\
&\le \|f\|_{C^\alpha} \int_{|x-y|\le 2\varepsilon} (|D_{ij}\Gamma| + \frac{2}{\varepsilon}|D_i\Gamma|)|x-y|^\alpha dy \\
&\le C\|f\|_{C^\alpha}(\int_{|x-y|\le 2\varepsilon} \frac{1}{|x-y|^{n-\alpha}}dy + \frac{2}{\varepsilon}\int_{|x-y|\le 2\varepsilon} \frac{1}{|x-y|^{n-1-\alpha}}dy) \\
&\le C\|f\|_{C^\alpha}((2\varepsilon)^\alpha + \frac{2}{\varepsilon}(2\varepsilon)^{1-\alpha}) \\
&\le C\|f\|_{C^\alpha}\varepsilon^\alpha.
\end{aligned}
$$

Thus $D_j v_\varepsilon(x)$ converges to u uniformly on compact subsets as $\varepsilon \to 0$. But $v_\varepsilon \to D_i\omega$, so $\omega \in C^2(\Omega)$ and $u = D_{ij}(\Omega)$.

Now take $\Omega_0 = B_R(0) \supset \Omega$. Since $\Delta\Gamma = 0$ away from 0, the integral formula tells us

$$
\begin{aligned}
\Delta\omega(x) &= -f(x)\int_{|x-y|=R} D_i\Gamma\nu_i(y)ds_y \\
&= \frac{f(x)}{n\omega_n}\int_{|x-y|=R} \frac{(x_i-y_i)}{|x-y|^n}\nu_i(y)dy \\
&= \frac{f(x)}{n\omega_n}\int_{|x-y|=R} \frac{(x_i-y_i)}{|x-y|^n}\frac{(x_i-y_i)}{|x-y|})dy \\
&= \frac{f(x)}{n\omega_n}\frac{1}{R^{n-1}}\int_{|x-y|=R} ds_y \\
&= f(x).
\end{aligned}
$$

■

Remark 1 *In fact, in the proof we only needed Dini continuity, i.e.*$|f(x)-f(y)| \le \varphi(|x-y|)$ *where* $\int_0^\infty \frac{\varphi(r)}{r}dr < \infty$.

$C^{2,\alpha}$ estimates for Poisson equation.

Proposition 2 *Consider* $B_1 = B_R(x_0) \subset B_{2R}(x_0) = B_2$, $f \in C^\alpha(B_{2R})$, *where* $0 < \alpha < 1$. *Let* $\omega(x) = \int_{B_2}\Gamma(x-y)f(y)dy$, *the Newtonian Potential of* f *in* B_2. *Then* $\omega \in C^{2,\alpha}(B_R(x_0))$ *and we have estimate*

$$\|D^2\omega\|_{0;B_R} + R^\alpha|D^2\omega|_{\alpha;B_R} \le C(\|f\|_{0;B_{2R}} + R^\alpha|f|_{\alpha;B_{2R}}),$$

where $C = C(n,\alpha)$ *is constant.*

Proof: Let $x_1 \in B_1$, in last proposition we have showed

$$D_{ij}\omega(x) = \int_{B_2} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j d\sigma_y.$$

Thus

$$\begin{aligned}
|D_{ij}\omega(x)| &\le |f|_{C^\alpha(x)} C\int_{B_2} |D_ij\Gamma(x-y)||x-y|^\alpha dy + |f(x)|\int_{\partial B_2} |D_i\Gamma(x-y)|ds_y \\
&\le C|f|_{C^\alpha(x)}\int_{B_2}\frac{1}{|x-y|^{n-\alpha}}dy + C|f(x)|\int_{\partial B_2}\frac{1}{|x-y|^{n-1}}ds_y \\
&\le C|f|_{C^\alpha(x)}\int_{B_{3R}(x)}\frac{1}{|x-y|^{n-\alpha}}dy + C|f(x)|\frac{1}{R^{n-1}}n\omega_n(2R)^{n-1} \\
&\le C|f|_{C^\alpha(x)}(3R)^\alpha + C|f(x)| \\
&\le C(|f(x)| + R^\alpha |f|_{C^\alpha(x)}).
\end{aligned}$$

Take $\overline{x} \in B_1$, then

$$D_{ij}\omega(\overline{x}) = \int_{B_2} D_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))dy - f(\overline{x})\int_{\partial B_2} D_i\Gamma(\overline{x}-y)\nu_j d\sigma_y,$$

Thus

$$\begin{aligned}
D_{ij}\omega(\overline{x}) - D_{ij}\omega(x) = &\int_{B_2} D_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))dy - \int_{B_2} D_{ij}\Gamma(x-y)(f(y)-f(x))dy \\
&- f(\overline{x})\int_{\partial B_2} D_i\Gamma(\overline{x}-y)\nu_j ds_y + f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y) ds_y.
\end{aligned}$$

Boundary terms are

$$f(x)\int_{\partial B_2}(D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y))\nu_j ds_y \qquad (I)$$

$$(f(x)-f(\overline{x}))\int_{\partial B_2}(D_i\Gamma(\overline{x}-y))\nu_j ds_y \qquad (II)$$

Solid terms in $B_\delta(\xi)$, where $\delta = |x-\overline{x}|, \xi = \frac{1}{2}(x+\overline{x})$, are

$$\int_{B_\delta(\xi)} D_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))dy \qquad (III)$$

$$\int_{B_\delta(\xi)} D_{ij}\Gamma(x-y)(f(x)-f(y))dy \qquad (IV)$$

On $B_2 \setminus B_\delta(\xi)$, we write $D_{ij}\Gamma(\overline{x}-y) = (D_{ij}\Gamma(\overline{x}-y) - D_{ij}\Gamma(x-y)) + D_{ij}\Gamma(x-y)$, the corresponding solid terms are

$$\int_{B_2\setminus B_\delta(\xi)}(D_{ij}\Gamma(\overline{x}-y) - D_{ij}\Gamma(x-y))(f(y)-f(\overline{x}))dy \qquad (V)$$

$$(f(x)-f(\overline{x}))\int_{B_2\setminus B_\delta(\xi)} D_{ij}\Gamma(x-y)dy \qquad (VI)$$

Now we begin to estimate these terms separately.

$$
\begin{aligned}
(I) &= f(x)\int_{\partial B_2}(D_i\Gamma(x-y)-D_i\Gamma(\overline{x}-y))\nu_j ds_y \\
&\le |f(x)|\int_{\partial B_2}|DD_i\Gamma(\hat{x}-y)||x-\hat{x}|ds_y \\
&\le |f(x)|\delta\cdot c\int_{\partial B_2}\frac{1}{|\hat{x}-y|^n}ds_y \le c|f(x)|\delta\frac{1}{R^n}n\omega_n(2R)^{n-1} \\
&\le c|f(x)|\delta\frac{1}{R} \le 2c|f(x)|\frac{\delta}{2R} \le 2c|f(x)|(\frac{\delta}{2R})^\alpha \\
&\le c|f(x)|\frac{\delta^\alpha}{R^\alpha}.
\end{aligned}
$$

$$
\begin{aligned}
(II) &= (f(x)-f(\overline{x}))\int_{\partial B_2}(D_i\Gamma(\overline{x}-y))\nu_j ds_y \\
&\le |f|_{C^\alpha(x)}\delta^\alpha c\int_{\partial B_2}\frac{1}{|\overline{x}-y|^{n-1}}ds_y \\
&\le |f|_{C^\alpha(x)}\delta^\alpha c\frac{1}{R^{n-1}}n\omega_n(2R)^{n-1} \\
&\le C|f|_{C^\alpha(x)}\delta^\alpha.
\end{aligned}
$$

$$
\begin{aligned}
(III) &= \int_{B_\delta(\xi)} D_{ij}\Gamma(\overline{x}-y)(f(y)-f(\overline{x}))dy \\
&\le \int_{B_\delta(\xi)}|D_{ij}\Gamma(\overline{x}-y)||(f(y)-f(\overline{x}))|dy \\
&\le c\int_{B_\delta(\xi)}\frac{c}{|\overline{x}-y|^n}|\overline{x}-y|^\alpha|f|_{C^\alpha(\overline{x})}dy \\
&\le c|f|_{C^\alpha(\overline{x})}\int_{B_\delta(\xi)}\frac{1}{|\overline{x}-y|^{n-\alpha}}dy \\
&\le c|f|_{C^\alpha(\overline{x})}\int_{B_{\frac{3\delta}{2}}(\overline{x})}\frac{1}{|\overline{x}-y|^{n-\alpha}}dy \\
&\le c|f|_{C^\alpha}(\frac{3\alpha}{2})^\alpha \\
&\le c|f|_{C^\alpha(\overline{x})}\delta^\alpha.
\end{aligned}
$$

Similarly,

$$
(IV) \le c|f|_{C^\alpha(x)}\delta^\alpha.
$$

The last two terms are to be continued.