18.175: Lecture 17

Poisson random variables

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Outline

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables

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Recall local CLT for walks on $\mathbb{Z}$

- Suppose $X \in b + h\mathbb{Z}$ a.s. for some fixed constants b and h.
- Observe that if $\phi_X(\lambda) = 1$ for some $\lambda \neq 0$ then X is supported on (some translation of) $(2\pi/\lambda)\mathbb{Z}$. If this holds for all λ, then X is a.s. some constant. When the former holds but not the latter (i.e., ϕ_X is periodic but not identically 1) we call X a **lattice random variable**.
- Write $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in \mathcal{L}_n := (nb + h\mathbb{Z})/\sqrt{n}$ and $n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$.
- Assume X_i are i.i.d. lattice with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. **Theorem:** As $n \to \infty$,

$$\Big| \sup_{x \in \mathcal{L}^n} \left| n^{1/2}/h p_n(x) - n(x) \right| \to 0.$$

Recall local CLT for walks on $\mathbb{Z}$

- **Proof idea:** Use characteristic functions, reduce to periodic integral problem. Look up "Fourier series". Note that for Y supported on $a + \theta\mathbb{Z}$, we have $P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \phi_Y(t) dt$.

Extending this idea to higher dimensions

- Example: suppose we have random walk on $\mathbb{Z}$ that at each step tosses fair 4-sided coin to decide whether to go 1 unit left, 1 unit right, 2 units left, or 2 units right?
- What is the probability that the walk is back at the origin after one step? Two steps? Three steps?
- Let's compute this in Mathematica by writing out the characteristic function ϕ_X for one-step increment X and calculating $\int_0^{2\pi} \phi_X^k(t)dt/2\pi$.
- How about a random walk on $\mathbb{Z}^2$?
- Can one use this to establish when a random walk on $\mathbb{Z}^d$ is recurrent versus transient?

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Poisson random variables: motivating questions

- How many raindrops hit a given square inch of sidewalk during a ten minute period?
- How many people fall down the stairs in a major city on a given day?
- How many plane crashes in a given year?
- How many radioactive particles emitted during a time period in which the expected number emitted is 5?
- How many calls to call center during a given minute?
- How many goals scored during a 90 minute soccer game?
- How many notable gaffes during 90 minute debate?
- **Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).

Bernoulli random variable with n large and $np = \lambda$

- Let λ be some moderate-sized number. Say $\lambda = 2$ or $\lambda = 3$. Let n be a huge number, say $n = 10^6$.
- Suppose I have a coin that comes up heads with probability λ/n and I toss it n times.
- How many heads do I expect to see?
- Answer: $np = \lambda$.
- Let k be some moderate sized number (say $k = 4$). What is the probability that I see exactly k heads?
- Binomial formula: $\binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} p^k (1-p)^{n-k}$.
- This is approximately $\frac{\lambda^k}{k!}(1-p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$.
- A **Poisson random variable** X with parameter λ satisfies $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$.

Probabilities sum to one

- A **Poisson random variable** X with parameter λ satisfies $p(k) = P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$.
- How can we show that $\sum_{k=0}^{\infty} p(k) = 1$?
- Use Taylor expansion $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.

Expectation

- A **Poisson random variable** X with parameter λ satisfies $P\{X=k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$.
- What is $E[X]$?
- We think of a Poisson random variable as being (roughly) a Bernoulli (n,p) random variable with n very large and $p = \lambda/n$.
- This would suggest $E[X] = \lambda$. Can we show this directly from the formula for $P\{X=k\}$?
- By definition of expectation

$$E[X] = \sum_{k=0}^{\infty} P\{X=k\}k = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}e^{-\lambda}.$$

- Setting $j = k-1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}e^{-\lambda} = \lambda$.

Variance

- Given $P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$, what is $\mathrm{Var}[X]$?
- Think of X as (roughly) a Bernoulli (n, p) random variable with n very large and $p = \lambda/n$.
- This suggests $\mathrm{Var}[X] \approx npq \approx \lambda$ (since $np \approx \lambda$ and $q = 1 - p \approx 1$). Can we show directly that $\mathrm{Var}[X] = \lambda$?
- Compute
$$E[X^2] = \sum_{k=0}^{\infty} P\{X = k\}k^2 = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}.$$
- Setting $j = k - 1$, this is
$$\lambda \left(\sum_{j=0}^{\infty} (j+1) \frac{\lambda^j}{j!} e^{-\lambda} \right) = \lambda E[X+1] = \lambda(\lambda+1).$$
- Then $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

Poisson convergence

- Idea: if we have lots of independent random events, each with very small probability to occur, and expected number to occur is λ, then total number that occur is roughly Poisson λ.
- **Theorem:** Let $X_{n,m}$ be independent $\{0,1\}$-valued random variables with $P(X_{n,m}=1)=p_{n,m}$. Suppose $\sum_{m=1}^{n} p_{n,m} \to \lambda$ and $\max_{1 \le m \le n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were Z is Poisson(λ).
- **Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.

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Recall continuity theorem

- **Strong continuity theorem:** If $\mu_n \implies \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to a measure μ with characteristic function ϕ.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi'_X(0) = 0$ and $\phi''_X(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and $L'_X(0) = -\phi'_X(0)/\phi_X(0) = 0$ and $L''_X = -(\phi''_X(0)\phi_X(0) - \phi'_X(0)^2)/\ \phi_X(0)^2 = 1$.
- If $V_n = n^{-1/2}\sum_{i=1}^n X_i$ where X_i are i.i.d. with law of X, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
- When we zoom in on a twice differentiable function near zero (scaling vertically by n and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.

Stable laws

- Question? Is it possible for something like a CLT to hold if X has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^n X_i$ for some a. Could the law of these guys converge to something non-Gaussian?
- What if the L_{V_n} converge to something else as we increase n, maybe to some other power of $|t|$ instead of $|t|^2$?
- The the appropriately normalized sum should be converge in law to something with characteristic function $e^{-|t|^\alpha}$ instead of $e^{-|t|^2}$.
- We already saw that this should work for Cauchy random variables. What's the characteristic function in that case?
- Let's look up stable distributions.

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.
- What random variables are infinitely divisible?
- Poisson, Cauchy, normal, stable, etc.
- Let's look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?
- More general constructions are possible via Lévy Khintchine representation.

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