18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

February 27, 2019

This is a reminder that the problem set is due on Monday, so we should start it soon. A few bonus problems were also added that are a bit more challenging.

Next Wednesday, we will discuss the problem set in class. Professor Postnikov is pretty lenient with late problem sets, but don't turn them in after we discuss the solutions.

Last time, we started discussing statistics on permutations. We defined $\text{inv}(w)$ to be the number of inversions and $\text{cyc}(w)$ to be the number of cycles, and we found some generating functions

$$\sum_{w \in S_n} q^{\text{inv}(w)} = [n]_q!, \sum_{w \in S_n} x^{\text{cyc}\, w} = x(x+1)\cdots(x+n-1).$$

Definition 1

Let a **descent** of a permutation $w = (w_1, \cdots, w_n)$ be an index $1 \le i \le n-1$ such that $w_i > w_{i+1}$. Denote the number of descents to be $\text{des}(w)$.

For example, $\text{des}(2,5,7,3,1,6,8,4) = 3$. The form of the generating function is a bit less nice, though.

Definition 2

The generating function

$$\sum_{w \in S_n} x^{\text{des}(w)}$$

is called the **Eulerian polynomial**.

So we have a number of inversions, cycles, and descents. Here's a meta-mathematical claim: interesting permutation statistics are likely equidistributed with one of these classes!

Fact 3

Things related to inversions are called "Mahonian statistics." Cycles are related to Stirling numbers, but there's no common name. Descent-related statistics are called "Eulerian statistics."

Definition 4

Let the **major index** of $w \in S_n$ be

$$\mathrm{maj}(w) = \sum_{\substack{i \text{ descent} \\ \text{of } w}} i.$$

For example, the permutation $w = (2, 5, 7, 3, 1, 6, 8, 4)$ has descents in position 3, 4, and 7, so $\mathrm{maj}(w) = 3+4+7 = 14$.

Theorem 5

$\mathrm{inv}(w)$ and $\mathrm{maj}(w)$ are equidistributed.

This is a bonus problem from the problem set! Both of these are named after Major Percy MacMahon, who wrote a famous book on combinatorics. So major is named that way for the military rank.

Definition 6

A **record** of a permutation w is an entry greater than all entries to the left. Define $\mathrm{rec}(w)$ to be the number of records of w.

For example, $w = (2, 5, 7, 3, 1, 6, 8, 4)$ has records $2, 5, 7, 8$, so $\mathrm{rec}(w) = 4$.

Theorem 7

$\mathrm{rec}(w)$ and $\mathrm{cyc}(w)$ are equidistributed.

This can be proved by induction: for example, show the generating functions satisfy the same recurrence relation. But we prefer combinatorial proofs: maybe we can find a transformation with a bijective argument!

Proof. We'll find some bijection $f : S_n \to S_n$ sending $w \to \tilde{w}$ such that $\mathrm{cyc}(w) = \mathrm{rec}(\tilde{w})$. Write permutations in cycle notation: if

$$w = (a_1 \cdots)(a_2 \cdots)(a_3 \cdots) \cdots,$$

there are many ways we can write this permutation down. Write w such that each a_i is the **maximal element** in its cycle, and sort them such that $a_1 < a_2 < \cdots$. Then $\tilde{w}$ is just w, but instead of viewing it as cycle notation, view it as one-line notation! For example,

$$w = (125)(3784)(6) \implies w = (512)(6)(8437) \implies \tilde{w} = (5, 1, 2, 6, 8, 4, 3, 7).$$

Notice that $a_1, a_2, \cdots$ will be the records of $\tilde{w}$, so the number of cycles of w is the number of records of $\tilde{w}$, as desired. This is bijective (to go backwards, find the records and put parentheses back), so we're done! ☐

Definition 8

An **exceedance** in w is an index $1 \le i \le n$ such that $w_i > i$. Let $\mathrm{exc}(w)$ be the number of exceedances of w.

For example, in $(2, 5, 7, 3, 1, 6, 8, 4)$, we have 4 exceedances (not counting 6, which is a "weak exceedance"), so $\mathrm{exc}(w) = 4$.

Theorem 9

exc(w) and des(w) are equidistributed.

In other words, "the number of exceedances is an Eulerian statistic."

Proof. Let's start with a related idea:

Definition 10

Define an **anti-exceedance** to be an index i such that $w_i < i$.

This is equidistributed with the number of exceedances. Why? Take the inverse permutation. If $w(i) > i$, then $w^{-1}(w(i)) < w(i)$: this means that i is an exceedance in w means $w(i)$ is an exceedance in w^{-1}.

Claim 10.1. Given a map $w \to \tilde{w}$ that converts cycle notation to one-line notation, the number of anti-exceedances in w is the number of descents in $\tilde{w}$.

This is because i being a descent in $\tilde{w}$ means that because the $i+1$th entry is not larger than $\tilde{w}$, i and $i+1$ are in the same cycle. Then that means that i goes to something smaller than itself, making it an anti-exceedance! $\square$

Let's look a bit at Stirling numbers now. There are two kinds of Stirling numbers: the first kind and the second kind.

Definition 11

Define the **Stirling numbers of the first kind** for $0 \le k \le n$

$$s(n,k) = (-1)^{n-k} c(n,k),$$

where $c(n,k)$ is the **signless Stirling numbers of the first kind**

$$c(n,k) = \text{the number of permutations} \in S_n \text{ with } k \text{ cycles.}$$

By convention, let $s(0,0) = 1$ and $s(n,0) = 0$ for all $n \ge 1$.

Here, fixed points are included as cycles (of length 1).

Fact 12

The generating function for the Stirling numbers

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1),$$

and we can equivalently find that

$$\sum_{k=0}^{n} s(n,k)x^k = x(x-1)\cdots(x-n+1).$$

The first expression is called "raising power of x," while the second is called the "falling power of x." The latter is sometimes denoted $(x)_n$.

Definition 13

Define the **Stirling number of the second kind**

$$S(n,k) = \text{number of set-partitions of } [n] \text{ into } k \text{ non-empty blocks.}$$

We also use the convention $S(0,0) = 1$ and $S(n,0) = 0$ for $n \geq 1$.

Example 14

One set-partition is $\pi = (125|3478|6)$. The main difference between this and cyclic notation is that the order within each group doesn't matter.

The Stirling numbers of the second kind are always positive: there's no negative signs like in the first kind.

Theorem 15

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$

Compare this to the generating function for Stirling numbers of the first kind: in that one, we input powers of x, and get falling powers of x, and in this one, we input falling powers of x and get powers of x! This is a kind of duality.