18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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We have one more presentation today:

Problem 1

Let $k \leq \frac{n}{2}$. Find a bijection f between k-element and $(n-k)$-element subsets of $[n]$ that $f(I) \supset I$, for any k-element subset I.

Solution by Chiu Yu Cheng. Note that the complement of a k-element subset has $n-k$ elements. Thus, it suffices to find a function $f' : I \to f'(I)$ such that $f'(I) \cap I = \varnothing$.

Put $1, 2, \cdots, n$ in a circle. Initialize $f'(I)$ to the empty set. For each $x \in I$, move clockwise until we meet the first element not in I and $f'(I)$ already, and put that in $f'(I)$.

We just need to show this is a bijection. To do this, we want to show that $f'(I)$ is determined regardless of the order of I. Given $x \notin I$, is it in $f'(I)$? Assign a number 1 to a number on the circle if it is in I and -1 otherwise. x is in $f'(I)$ if and only if there is a counterclockwise partial sum starting from $x-1$ that is positive. This is independent of the order of I chosen! □

This was a Google problem for getting an interview a while ago. We'll have another solution to this problem as we talk about some more concepts!

Remember the ideas of posets and lattices from a few lectures ago: we have a set L with operations $\wedge$ and $\vee$. Alternatively, we can define an operation $\leq$, where $\wedge$ (meet) is the unique maximal element $\leq$ both x and y, and $\vee$ (join) is the unique minimal element such that both x and y are $\leq$ it. It can be shown that this poset chain satisfies the axioms that we want: the idea is that

$$x \leq y \iff x \wedge y = x.$$

An axiom of the lattice definition is that $x \wedge y = x \iff x \vee y = y$, so this is consistent.

Definition 2

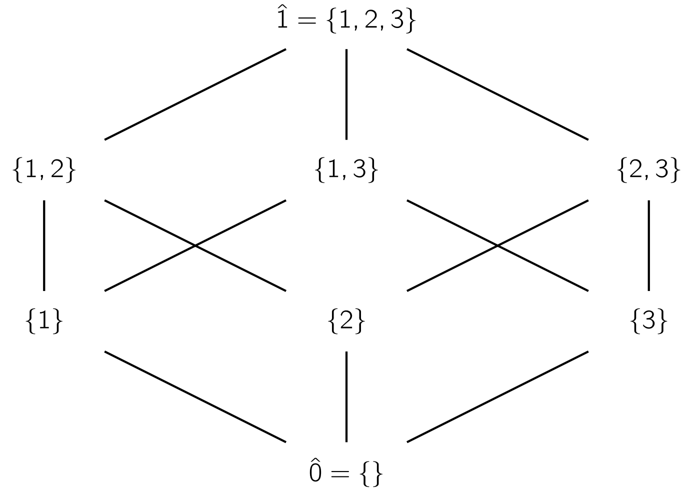
The **Boolean lattice** B_n is defined with elements that are subsets of $\{1, \cdots, n\}$. Our order relation $S_1 \leq S_2 \iff S_1 \subset S_2$. Then $\wedge$ and $\vee$ have nice interpretations:

$$S_1 \wedge S_2 = S_1 \cap S_2, S_1 \vee S_2 = S_1 \cup S_2.$$

This might explain why $\wedge$ and $\vee$ look the way they do!

Example 3

The Boolean lattice B_3 looks like this, where $\hat{0}$ denotes the smallest element and $\hat{1}$ denotes the largest element:



Notice that this traces out a 3-dimensional cube!

Definition 4

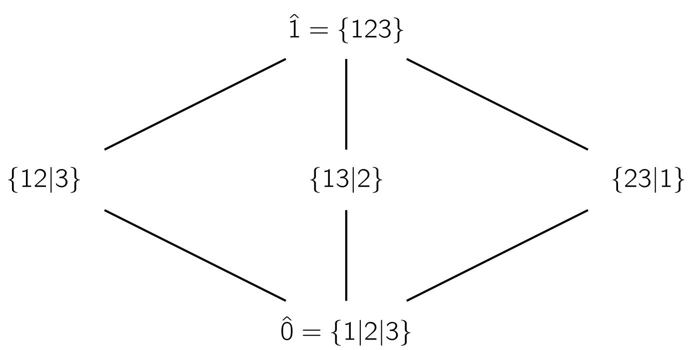
The **partition lattice** Π_n has elements that are set-partitions of $\{1, \cdots, n\}$. They are ordered by **refinement**: $\pi \leq \sigma$ (which means π **refines** σ or σ **coarsens** π) if each block of π is contained in a block of σ.

Here, the meet operation $\sigma \wedge \pi$ is the **common refinement** of σ and π: take all intersections of blocks.

On the other hand, the join operation $\sigma \vee \pi$ is the **finest common coarsening** of σ and π, but this is slightly harder to define. If a, a' are in the same block of π, and a', a'' are in the same block of σ, a'', a''' are in the same block of π, and so on (alternating between π, σ), then those elements along the chain are in the same block of $\sigma \vee \pi$.

Example 5

Here's what the partition lattice Π_3 looks like:



Definition 6

Young's lattice $\mathbb{Y}$ is an infinite lattice of all Young diagrams ordered by containment.

The minimal element $\hat{0}$ is the empty Young diagram. Above it, we have the diagram with a single box, then 2 dominoes, and then there are 3 Young diagrams of 3 boxes, 5 Young diagrams of 4 boxes, and so on.

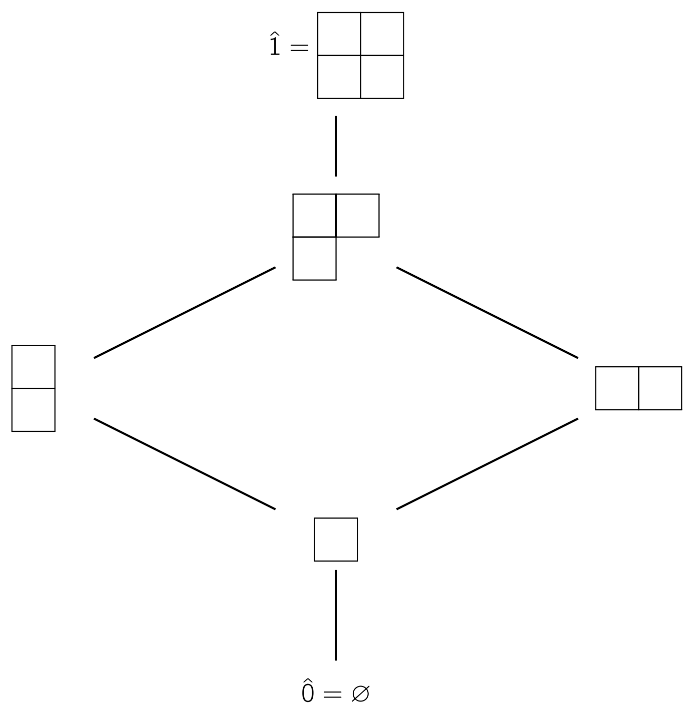
Fact 7

Given any Young diagram, there is always 1 more shape above it than there is below it (think about the corners)!

There are finite sublattices of $\mathbb{Y}$: for example, we can fix m, n, and define $L(m, n)$ to be the sublattice of Young diagrams that fit inside an $m \times n$ box. Recall that those are generated by the generating function $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Example 8

$L(2, 2)$ is a finite lattice: it has a unique maximal and minimal element. It looks like this:



What are the meet and join operations? Here, $\lambda \wedge \mu$ is the set-theoretic intersection, and $\lambda \vee \mu$ is the union. It can be checked that these are indeed always Young diagrams!

Among all lattices, there is a special class:

Definition 9

A lattice $(L, \wedge, \vee)$ is **distributive** if it satisfies the distributive laws

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Notice that this is not satisfied by normal operations: $x + (yz) \neq (x + y)(x + z)$.

Lemma 10

B_n is a distributive lattice!

Proof. It is easy to check that in general, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, and similarly $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$. (Draw a Venn diagram!) □

Lemma 11

The Young lattice $\mathbb{Y}$, as well as $L(m, n)$ are distributive lattices.

Proof. Meet and join are still just unions and intersections here! So the same proof works. □

Fact 12

Unfortunately, Π_n for $n \geq 3$ is not a distributive lattice. So not all lattices are distributive! For example, let $x = (12|3), y = (13|2), z = (23|1)$. Then

$$x \vee (y \wedge z) = x, (x \wedge y) \vee (x \wedge z) = \hat{1}.$$

Turns out there is a very simple description of finite distributive lattices:

Definition 13

Given a poset P, $I \subset P$ is an **order ideal** if for all $x \in I, y \leq x$, $y \in I$.

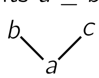
So ideals basically contain some "bottom part" of the Hasse diagram: it's "closed downward."

Definition 14

Given a poset P, denote $J(P)$ to be the poset of all order ideals in P, ordered by containment.

Theorem 15 (Birkhoff's FTFDL (Fundamental Theorem for Finite Distributive Lattices))

The map $P \to J(P)$ is a one-to-one correspondence between finite posets and finite distributive lattices!

For example, if our poset has elements $a \leq b, a \leq c$, then the order ideals are $\{\}, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$. In other words, here's the poset $J(P)$ for $P =$ 

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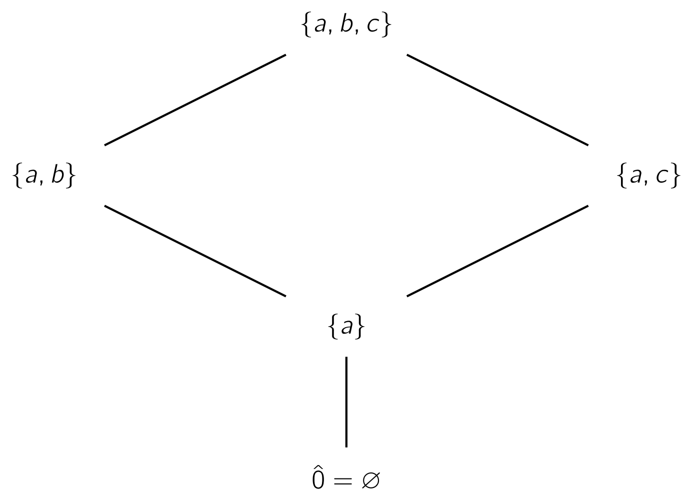

We'll look a bit more at this next time!