18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

April 3, 2019

Recall that last lecture, we formulated the Jacobi Triple Product identity, which can be written equivalently in the following form:

$$\prod_{n\geq 1}(1+zq^n)\prod_{n\geq 1}(1+z^{-1}q^{n-1}) = \sum_{r=-\infty}^{\infty} z^r q^{r(r+1)/2} \prod_{n\geq 1}\frac{1}{1-q^n}.$$

Let's do a combinatorial proof of this identity! Each of the products has a combinatorial interpretation in terms of certain partitions.

Proof. First of all, the first product's coefficient of z^a is

$$\sum_{\substack{\mu \text{ partition} \\ \text{with } a \text{ distinct parts}}} q^{|\mu|},$$

and the second product's coefficient of z^{-b} is

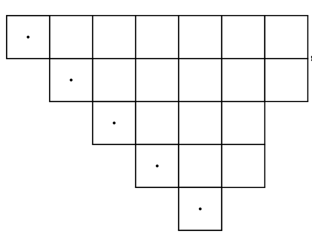
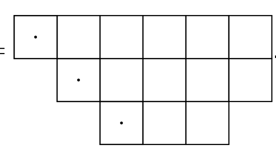
$$\sum_{\substack{\nu \text{ partition} \\ \text{with } b \text{ distinct parts}}} q^{|\nu|-b|},$$

since the power is $n-1$, not n. Finally, the product on the right is the usual generating function for all partitions: somehow we want to combine different partitions together. So we want a bijection between

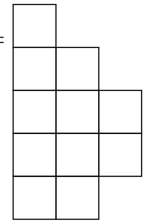
$$(a, b, \mu, \nu) \to (r, \lambda).$$

where μ has a distinct parts and ν has b distinct parts. We also need to make sure the monomials match up: looking at powers of z, $\boxed{a-b=r}$, and looking at powers of q, $\boxed{|\mu|+|\nu|-b = \frac{r(r+1)}{2}+|\lambda|}$.

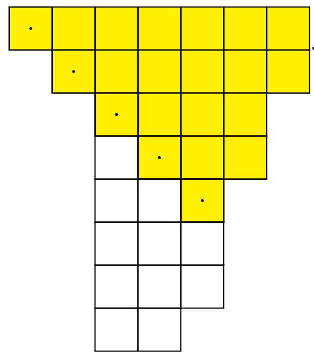
Since μ and ν have distinct parts, we can represent them with shifted Young diagrams instead! For example, if $\mu = (7, 6, 4, 3, 1)$ and $\nu = (6, 5, 3)$, then our Young diagrams look like

$\tilde{\mu} =$ , $\tilde{\nu} =$ .

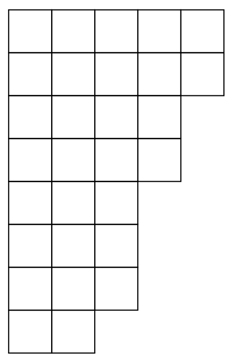
We're going to transpose $\tilde{\nu}$ and remove its b diagonal boxes:

$\tilde{\nu}' =$ 

Now, take $\tilde{\mu}$ and $\tilde{\nu}'$ and glue their last dots together! This now looks like

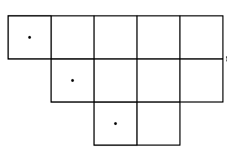
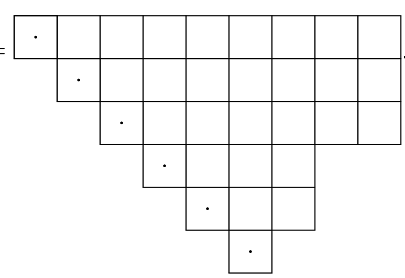
.

With this, we can now chop off the first two columns off: our partition λ is now

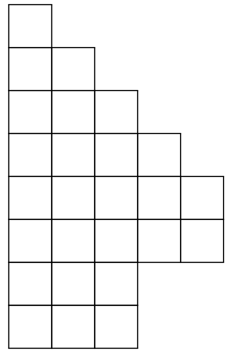
$\lambda =$  $= (5, 5, 4, 4, 3, 3, 3, 2)$.

We claim that this is the bijection that we want! The number of columns we chop off is $a - b = r$, and counting the total number of boxes, we started with $|\mu| + |\nu|$ boxes, but then we removed all b diagonal boxes from ν: this yields the left hand side. That left us with $|\lambda| + 1 + 2 + \cdots + r$ boxes, which is the desired right hand side!

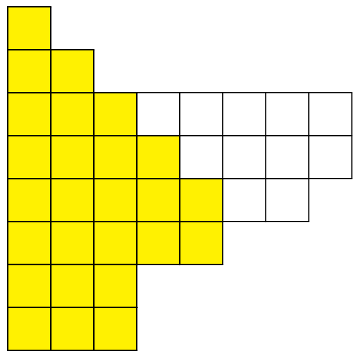
We might not be convinced: what if $a < b$? We have to make a small edit, and we'll do that by construction. If $\mu = (5, 4, 2)$ and $\nu = (9, 8, 7, 4, 3, 1)$, then our shifted Young diagrams look like

$\tilde{\mu} =$ , $\tilde{\nu} =$ .

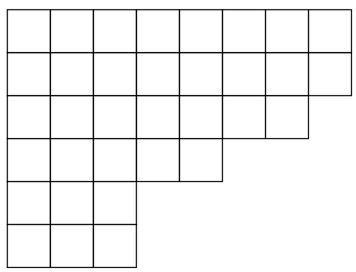
Transposing and removing boxes, we have

$\tilde{\nu}' =$ 

and after gluing, our shape is



which yields (after chopping)

$\lambda =$  $= (8, 8, 7, 5, 3, 3).$

We need to make sure the identity still works: the total number of boxes here is $|\mu| + |\nu| - b$, but now the size of the triangle is $b - a - 1$ (since our triangle is now below the dots)! So we have a small triangle that we chopped off, which is $\frac{(-r-1)(-r)}{2} = \frac{r(r+1)}{2}$ squares, so the same identity holds.

Why is this a bijection? We need to know how to invert this from λ and r: attach a small triangle to the left or top of λ based on the sign of r, and then we chop along the diagonal. This gives us $\tilde{\mu}$ and $\tilde{\nu}'$, which can give us μ and ν, which tells us our original partitions and a and b as well! $\square$

Let's move on to the next topic of this class. We've been talking about Young diagrams and partitions a lot so far - in this second half, we'll start to discuss graphs, networks, and trees! (Graphs are objects with vertices and edges.)

Definition 1

A **labeled tree** is a connected graph on n nodes with no cycles, where the nodes are labeled $1, 2, \cdots, n$. **Unlabeled trees** are the analogous object where the nodes are not labeled.

Example 2

There are 3 labeled trees on 3 vertices: put $1, 2, 3$ at the vertices of an equilateral triangle, and remove one of the edges. However, there is only 1 unlabeled tree on 3 vertices: it is a path.

Turns out we have an explicit formula for the number of trees:

Theorem 3 (Cayley's formula)

The number of labeled trees on n nodes is n^{n-2}.

These numbers n^{n-2} also appear pretty often! This is probably the next most famous set of numbers after Catalan numbers, and there are many combinatorial interpretations.

Fact 4

In mathematics, there's a law that mathematical formulas are never named after the person who discovered it. Cayley wrote a paper in 1889, but the same result was given by Borchord in 1860 and Sylvester in 1857.

Cayley didn't actually give a complete proof: it's not a very trivial formula, but Professor Postnikov wants to show his favorite proof, which is probably the shortest one!

Proof by Rényi, 1967. We're going to use induction (?!?!). It's kind of hard to get n^{n-2} from smaller versions of itself, though... let's do a generalization! Let's prove the more general result:

Definition 5

Define the **weight** of a tree T to be a monomial $x^T \equiv x_1^{\deg(1)-1} x_2^{\deg(2)-1} \cdots x_n^{\deg(n)-1}$.

Notice that leaves have exponent 0. Now define

$$F_n(x_1, \cdots, x_n) = \sum_{\substack{\text{labeled trees on} \\ n \text{ vertices}}} x^T.$$

We claim that this quantity is equal to $(x_1 + \cdots x_n)^{n-2}$, and notice that Cayley's formula follows by setting all $x_i = 1$. The idea is that the inductive hypothesis is now stronger, so we can actually use induction! Define $R_n = F_n - (x_1 + \cdots + x_n)^{n-2}$; our goal is to show that $R_n = 0$.

This is easy to check for base cases $n = 1, 2$. Now let's make some observations:

- R_n is a polynomial in $x_1, \cdots, x_n$ of degree $\le n-2$. This is because the second term has degree $n-2$, and every monomial has degree $n-2$: there are $n-1$ edges, and each one contributes to 2 exponents (the endpoints of the edge), and the -1s in the definition of x^T subtract n, for a total degree of $2(n-1) - n = n-2$.
- For any $i = 1, 2, \cdots, n$, if we initialize $x_i = 0$, R_n evaluates to 0. To prove this, we can assume $i = n$ by symmetry: set $x_n = 0$, and now all x^T die except for those where the degree of n is 1: that is, n is a leaf! So F_n evaluated at $x_n = 0$ is
$$\sum_{T: n \text{ leaf}} x^T = \sum_{T' \text{ with } n-1 \text{ nodes}} x^{T'}(x_1 + x_2 + \cdots + x_{n-1})$$
since there are $n-1$ ways to connect n to one of the other vertices. By the induction hypothesis, this is $F_{n-1}(x_1 + \cdots + x_{n-1})$, so $R_n|_{x_n=0} = R_{n-1}(x_1 + \cdots + x_{n-1}) = 0$.
- But now we're done: if $f(x_1, \cdots, x_n)$ is a polynomial on n variables of degree at most $n-1$, and $x_1, x_2, \cdots, x_n$ are all factors, then the polynomial must be 0.

$\square$

This almost feels like cheating! We used almost no properties of trees at all. We'll talk a bit more about this next time!