18.212: Algebraic Combinatorics

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Spring 2019

This class is being taught by **Professor Postnikov**.

April 26, 2019

Today we will continue with high school physics! Just to recap, let G be a connected digraph - directions don't actually matter, but they are used for convenience, and we select two vertices A and B to be connected to a battery. Then we have some current I through the battery, and our goal is to find all the relevant currents and voltages I_e, V_e given the resistances R_e for each edge. Alternatively, how can we find the resistance of the whole electrical network R_{AB}?

First of all, remember that we have three laws that govern electricity. Let's try to write them out as matrix laws!

- Kirchhoff's second law says that we can define a potential function U on the vertices of our graph such that

$$V_e = U_v - U_u$$

  for any edge $e : u \to v$.

- Ohm's law gives us a proportionality relationship:

$$I_e = \frac{V_e}{R_e} = \frac{U_v - U_u}{R_e}.$$

- For any vertex v, the in-current is equal to the out-current. For example, if $e_1 : v_1 \to v$ and $e_2 : v_2 \to v$ lead into v, and e_3, e_4, e_5 lead out of v (from v to v_3, v_4, v_5 respectively), we can write that as

$$\frac{U_v - U_{v_1}}{R_1} + \frac{U_v - U_{v_2}}{R_2} = \frac{U_{v_3} - U_v}{R_3} + \frac{U_{v_4} - U_v}{R_4} + \frac{U_{v_5} - U_v}{R_5}.$$

For this last bullet point, let's instead move all terms to the left hand side: now we get the nice symmetric form

$$\sum_{i \text{ neighbor}} \frac{U_v - U_{v_i}}{R_i} = 0.$$

This isn't quite true always, though: we get some additional current if we're at A or B. We can instead write this as

$$\boxed{\left(\frac{1}{R_1} + \cdots + \frac{1}{R_d}\right) U_v - \sum_{i=1}^{d} \frac{1}{R_i} U_{v_i} = \begin{cases} 0 & v \neq A, B \\ -I & v = A \\ I & v = B \end{cases}}$$

Now let's label our vertices with the integers from 1 to n, such that $A = 1$ is the first one and $B = n$ is the last one. Denote R_{ij} to be the resistance of edge (i, j), and since we have reciprocals in our equation, define C_{ij} to be $\frac{1}{R_{ij}}$ if (i, j) is an edge of our graph and 0 otherwise. (Think of this as non-edges having infinite resistance!)

Now we can write all three electrical laws in terms of this matrix! Specifically, if $\vec{u} = \begin{bmatrix} U_1 \\ \vdots \\ U_N \end{bmatrix}$ is the vector of potentials,

we have

$$KU = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}. \tag{1}$$

Notice that the Kirchhoff matrix is really just the Laplacian matrix for graphs whose edge-weights are conductances!

Note that this system has many solutions, since all rows of our Kirchhoff matrix add to 0. But that makes sense: our potential functions can be shifted up and down by a constant! This is because K, our Laplacian or Kirchhoff matrix, has rank $n - 1$ if G is connected - this means that we have at least one spanning tree by the matrix tree theorem, so all cofactors have to be nonzero.

In particular, can we use this to calculate the resistance between A and B

$$R_{AB}(G) = \frac{U_n - U_1}{I}?$$

Let's try to calculate this more explicitly. Note that we can simultaneously rescale the U vector and I in our equation (1) above without changing K: let's assume $I = 1$, and then we can also assume by adding our constant that $U_1 = 0$ (by grounding the first vertex). Then the resistance of the whole circuit is just U_n. (1) then becomes

$$K^{(1)} \begin{bmatrix} U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

where $K^{(1)}$ denotes the $(n-1) \times (n-1)$ matrix obtained by removing the first row and column of K.

This is a matrix equation, and we should use Cramer's rule (despite professor Strang in 18.06 saying to never do so)! Since our goal is to calculate U_n,

$$R_{AB}(G) = U_n = \frac{\det \tilde{K}^{(1)}}{\det K^{(1)}},$$

where $\tilde{K}^{(1)}$ is obtained by replacing the last column with the vector $\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$. Notice that expanding this determinant along

the last column is actually just going to give us the determinant of the upper left $(n-2) \times (n-2)$ square of $K^{(1)}$! Denote this as $K^{(1,n)}$, where we remove the first and last rows and columns of our matrix.

The bottom line is that we get the following:

2

Theorem 3

The overall resistance of our graph G

$$R_{AB}(G) = \frac{\det K^{(1,n)}}{\det K^{(1)}} = \frac{\sum_{\substack{\tilde{T} \text{ spanning} \\ \text{tree of } \tilde{G}}} \text{weight}(\tilde{T})}{\sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \text{weight}(T)},$$

where the weight of a tree is the product of the conductances

$$\text{weight}(T) = \prod_{e \text{ edge of } T} \frac{1}{R_e},$$

and $\tilde{G}$ is obtained by gluing edges A and B together (or by connecting the two with an edge of infinite conductance).

Equivalently, we can think of the numerator a little differently: a spanning tree $\tilde{T}$ of $\tilde{G}$ is a forest of two components, such that A and B are in different components, so that the two are joined together when A and B are glued together.

Let's do an example!

Example 4

Consider a graph with three vertices A, B, C, where the resistances are R_1 between A and C, R_2 between B and C, and R_3 between A and B.

Then our theorem says that the total resistance between A and B is governed by spanning trees! $\tilde{G}$ makes A and B the same edge, so it only contains the edges R_1 and R_2: either of those is a spanning tree, so

$$R_{AB}(G) = \frac{(R_1)^{-1} + (R_2)^{-1}}{(R_1 R_2)^{-1} + (R_1 R_3)^{-1} + (R_2 R_3)^{-1}}.$$

We could have also done this by using series and parallel connections:

Proposition 5 (Series connection)

If we have two graphs G_1 and G_2, only connected by one vertex C, and $A \in G_1, B \in G_2$, then

$$R_{AB} = R_{AC} + R_{BC};$$

this can be also stated as $R(G_1 + G_2) = R(G_1) + R(G_2)$.

Proposition 6 (Parallel connection)

If we have two graphs G_1 and G_2 that are disjoint but share two vertices A and B, then

$$R(G_1 \parallel G_2)^{-1} = R(G_1)^{-1} + R(G_2)^{-1}.$$

Using these two processes, we can generate many graphs, but not all of them!

Fact 7

The smallest graph that doesn't work is the **Wheatstone bridge**: this is a 4-cycle with one diagonal. Then finding the resistance between the two other vertices can't be directly found by series and parallel connections.

That's an example where the general formula helps! By the way, we can easily deduce the series and parallel laws from the electrical network theorem, but we might come back to that later.

Now let's look at something a bit different: this time, if we're trying to work with our graph from A to B, rescale the currents and potentials so that $U_A = 0, U_B = 1$: all other vertices have potentials between 0 and 1. Can we phrase this as a probability?!

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Proposition 8

Consider a random walk on the vertices of G, where the probability of us jumping from u to v is

$$\Pr(u, v) = \frac{\frac{1}{R_{uv}}}{\sum_{w \neq u} \frac{1}{R_{uw}}},$$

where the denominator is a normalizing factor. Then the probability that a random walk starting at vertex v hits B before it hits A is U_v.

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This should look a lot like the drunk man problem from the first class! We can think of A as the point $x = 0$, B as the house at $x = N$, and then U is the probability that the man survives.

Proof. Let P_v be the probability of survival: writing out the linear equations for the random walk gives you the same equations as Kirchhoff's law! □

We'll continue with the **best theorem** next time.

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