18.212: Algebraic Combinatorics

Andrew Lin

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This class is being taught by **Professor Postnikov**.

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We'll talk some more about plane partitions today! Remember that we take an $m \times n$ rectangle, where we have m columns and n rows. We fill the grid with weakly decreasing numbers from 0 to k, inclusive.

Then the number of possible plane partitions, by the Lindstrom lemma, is the determinant of the $k \times k$ matrix C where entries are

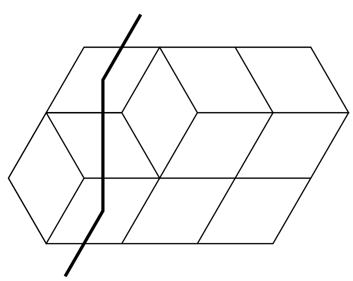
$$c_{ij} = \binom{m+n}{m+i-j}.$$

MacMahon also gave us the formula

$$= \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{\ell=1}^{k} \frac{i+j+\ell-1}{i+j+\ell-2}.$$

There are several geometric ways to think about this: plane partitions correspond to rhombus tilings of hexagons with side lengths m, n, k, m, n, k, as well as perfect matchings in a honeycomb graph. In fact, there's another way to think about these plane partitions: we can also treat them as **pseudo-line arrangements**.

Basically, pseudo-lines are like lines, but they don't have to be straight. We can convert any rhombus tiling into a pseudo-line diagram! Here's the process: start on one side of the region. For each pair of opposite sides of our hexagon, draw pseudo-lines by following the common edges of rhombi! For example, if we're drawing pseudo-lines from top to bottom, start with a rhombus that has a horizontal edge along the top edge of the hexagon, and then connect this rhombus to the rhombus that shares the bottom horizontal edge. Repeat until we reach the bottom edge of our hexagon. Here's an example of a pseudo-line:



Then we'll have some intersections between the pseudo-lines we've drawn, and this is in one-to-one correspondence with our original tiling! This comes from the theory of symmetric polynomials, which are related to Young diagrams and the Schensted correspondence, among many other things we've discussed in this class!

Definition 1

A polynomial $f(x_1, \cdots, x_n)$ is **symmetric** if it is invariant under the permutations of the variables: $f(x_1, \cdots, x_n) = f(x_{w(1)}, \cdots, x_{w(n)})$ for all permutations $w \in S_n$.

The simplest polynomials are called the **elementary symmetric polynomials** $e_1, \cdots, e_n$: we have

$$e_1 = x_1 + \cdots + x_n,$$

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,$$

$$\vdots$$

and in general, e_k is the sum of all square-free monomials of degree k. We have an important result:

Theorem 2

Any symmetric polynomial $f(x_1, \cdots, x_n)$ can be written uniquely as a polynomial in the elementary symmetric polynomials $e_1, \cdots e_n$.

To show why this is useful, we'll introduce a new combinatorial object:

Definition 3

A **semi-standard Young tableau** (SSYT) for a Young diagram λ is a filling of λ with the numbers $1, 2, \cdots$ such that the entries increase **strictly** in columns and **weakly** in rows.

Remember that standard Young tableau require us to use each number exactly once; this is no longer true here! Here's an example:

1	1	1	1	2	2	2	3
2	2	2	3	3	4	4	
3	3	3	4	4			
4	4	6	6				

Definition 4

Let λ be a Young diagram with at most n rows. Define the **Schur polynomial**

$$s_\lambda(x_1, \cdots, x_n) = \sum_{\substack{\text{SSYT of} \\ \text{shape } \lambda \\ \text{filled with} \\ [1,2,\cdots,n]}} x_1^{\#1\text{'s}} \cdots x_n^{\#n\text{'s}}$$

(Note that we need at most n rows, because the first entries of the rows are $1, 2, \cdots$, and so on.)

Example 5

The Schur polynomial for a single box is just $x_1 + x_2 + \cdots + x_n = e_1$, and in general, the polynomial for a column of k boxes is the kth symmetric polynomial, since we pick some k of the n numbers to put in our semistandard Young tableau.

When we add multiple columns, though, things become more interesting!

Example 6

The Schur polynomial for a horizontal domino is now $\sum_{i \leq j} x_i x_j = x_1^2 + x_1x_2 + x_2^2 + \cdots = e_1^2 - e_2$.

Theorem 7

Any symmetric polynomial $f(x_1, \cdots, x_n)$ can actually be written uniquely as a linear combination

$$f = \sum_\lambda c_\lambda s_\lambda(x_1, \cdots, x_n)$$

where s_λ is a Schur polynomial.

In other words, the Schur polynomials form a **linear basis**, and the elementary polynomials form an **algebraic basis**. How are these two theorems related to each other?

Theorem 8 (Jacobi-Trudi)

Let $\lambda = (\lambda_1, \cdots, \lambda_e)$ be a partition, where $e \leq n$, and let $\lambda' = (\lambda'_1, \cdots, \lambda'_m)$ be its **conjugate partition** (basically, we take the size of the columns instead of the size of the rows). Then

$$s_\lambda(x_1, \cdots, x_n) = \det \begin{pmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+(m-1)} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+(m-2)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_m-(m-1)} & e_{\lambda'_m-(m-2)} & \cdots & e_{\lambda'_m} \end{pmatrix}$$

where we assume $e_0 = 1$ and $e_{-k} = 0$ for all $k < 0$.

For example, the Schur polynomial of the horizontal domino, where $\lambda = (2)$ and $\lambda' = (1, 1)$, is

$$\det \begin{pmatrix} e_1 & e_2 \\ e_0 & e_1 \end{pmatrix}$$

This looks a lot like the binomial coefficients we got from the Lindstrom lemma, and indeed it's possible to connect these ideas! We can think of semistandard Young Tableau as noncrossing lattice paths, and this will indeed prove Jacobi-Trudi from the Lindstrom lemma.

But how are these connected to semistandard Young tableaux? For example, what if we want to calculate the number of them?

Theorem 9

We can just plug in $x_1 = x_2 = \cdots = x_n$: this yields

$$s_\lambda(1, 1, \cdots, 1) = \det C,$$

where the entries of our $m \times m$ matrix

$$c_{ij} = \binom{n}{\lambda'_i + j - i}.$$

It turns out that there are two ways to write this determinant as a product: it's equal to

$$= \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{a \in \lambda} \frac{n + c(a)}{h(a)}.$$

The latter is called the **hook-content formula**: here, $h(a)$ is the same **hook length** as we've been discussing, and $c(a)$ is the **content**, equal to $j - i$.

How is all of this related to plane partitions? We'll connect semi-standard Young tableaux with **reverse plane partitions**, which are those where the entries are **increasing** in rows and columns. If our SSYTs are filled with numbers from 1 to n, and our reverse plane partitions are filled with numbers from 0 to k, we have an exact product formula for **any** shape for Young tableaux but only really for **rectangular** plane partitions. Why are rectangular shapes better for reverse plane partitions?

Lemma 10

RPP's of rectangular shape $n \times m$ filled with numbers $0, 1, \cdots, k$ are in bijection with semistandard Young tableaux of the same shape $n \times m$ filled with $1, 2, \cdots, k+n$: add i to all entries in the ith row.

But notice that this bijection doesn't work for any other shapes! Rectangles only have one outer corner, and it's enough to say that the bottom right corner is at most k to force the numbers to be between 0 and k. Unfortunately, with nonrectangles, the multiple corners mess things up, and that means we don't actually have nice explicit formulas for other shapes of RPPs.

There are many books we can read and many other classes to learn about algebraic combinatorics