18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Recall the Schensted correspondence between permutations $w \in S_n$ and pairs of Young tableaux (P, Q): for example, we found that $w = (3, 5, 2, 4, 7, 1, 6)$ corresponds to

$$P = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

The process we took to get to P is

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \to P.$$

According to Schensted's theorem, the shape $\lambda = (3, 3, 1)$ tells how many boxes are in each row, and λ_1, the number of boxes in the first row, is the size of the longest increasing subsequence in w. Similarly, the shape $\lambda' = (3, 2, 2)$ tells us how many boxes are in each column, and λ'_1, the number of boxes in the first column, is the size of the longest decreasing subseqence in w.

We're going to prove the first half of this (increasing subsequence) and leave the other half as an exercise!

Definition 1

The ***j*th basic subsequence** in a permutation w, where $1 \le j \le \lambda_1$, consists of all entries of w that were originally inserted in the jth row.

For example, we have 3 basic subsequences for the example above. For B_1, note that we inserted 3, 2, and 1, so $B_1 = (3, 2, 1)$. Similarly, $B_2 = (5, 4)$, and $B_3 = (7, 6)$.

Lemma 2

Each B_j is a decreasing sequence.

Proof. By construction, if something bumps N, only smaller numbers can do this. So any number after N must be smaller. $\square$

Lemma 3

For all $j \geq 2$, given any $x \in B_j$, we can find $y \in B_{j-1}$ such that $y < x$ and y is located to the left of x in the permutation w.

Proof. At the moment of insertion of x, take y to be the entry that is located to the left of x. It will be less than x, and it was already inserted, so it is appears before x in the permutation. □

So now it's time to prove the Schensted theorem (part 1). λ_1 is the number of basic subsequences by definition. Note that given any increasing subsequence $x_1 < x_2 < \cdots < x_r$, we can only have at most 1 entry from each B_j. So this means $r \leq \lambda_1$. To construct an example of the equality case, pick the last basic subsequence B_{λ_1}, and pick any $x_{\lambda_1} \in B_{\lambda_1}$. By lemma 2, and we get an $x_{\lambda_1 - 1} \in B_{\lambda_1 - 1}$, and so on. Eventually we'll be done and have an increasing subsequence of length λ_1!

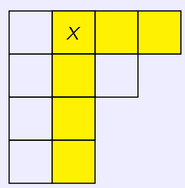
For the rest of today, we're going to prove the Hook Length Formula. Recall the theorem:

Theorem 4 (Hook Length Formula)

The number of ways to fill out a standard Young tableau with $|\lambda| = n$ is

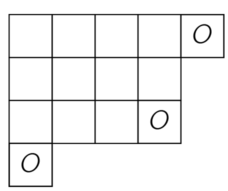
$$f^\lambda = \frac{n!}{H(\lambda)} = \frac{n!}{\prod_x h(x)}$$

where $h(x)$ is the "hook length" of x. For example, in the below diagram, the hook length $h(x) = 6$.



The original proof was pretty complicated, and it follows from some other formulas. But later, simpler proofs were found, and we're going to use a random process!

Hook walk proof by Greene, Nijenhuis, Wilf (1979). There is a recurrence relation for the number of Standard Young Tableau. Given a tableau with n boxes, n must appear in one of the bottom-right corner boxes, and removing this corner, we get a standard Young tableau of size $n - 1$. For instance, look at the following:



Since the O's are the spots that the largest number can be in,

$$f^\lambda = f^{4441} + f^{5431} + f^{544}.$$

So we'll try to prove the theorem by induction:

$$f^\lambda = \sum_{v \text{ corner of } \lambda} f^{\lambda - v}.$$

Base case: it's easy to prove this for $n = 0$ or 1.

Inductive step: Now we need to show that the same recurrence relation holds for the expression on the right hand side! In other words, we want to show that

$$\frac{n!}{\prod h(\lambda)} = \sum_{v \text{ corner of } x} \frac{(n-1)!}{H(\lambda - v)}.$$

This is the same as wanting to show that (dividing through by the left hand side)

$$1 = \sum_{v \text{ corner}} \frac{1}{n} \frac{H(\lambda)}{H(\lambda - v)}.$$

We have a 1 on the left side, and we have a bunch of nonnegative integers on the right side. So we can think of this as a probability distribution! We want to construct a random process so that these are probabilities.

Pick any box of λ uniformly at random; call it u. At each step, we can jump from u to any other square in the hook of u with equal probability. Repeat this process repeatedly, and stop once we reach a corner v.

Define $p(v)$ to be the probability that a hook-walk does end at corner v. We claim that the probability $p(v) = \frac{1}{n}\frac{H(\lambda)}{H(\lambda - v)}$, which means we would be done!

Why is this? Denote $P(u, v)$ to be the probability that a hook walk $(u, u', u'', \cdots, v)$ starting at box u ends at corner v. This is just summing over all hook walks:

$$P(u, v) = \sum_{u \to u' \to \cdots \to v} P(\text{this hook walk}) = \sum_{u \to u' \to \cdots \to v} \left(\frac{1}{h(u) - 1} \cdot \frac{1}{h(u') - 1} \cdots \right).$$

Here's the key observation: when we fix v, the whole hook-walk stays within the rectangle between the top-left corner and v. Whenever we have a rectangle with corners a, b, c, d (a in the top left and d a corner), $h(a) + h(d) = h(b) + h(c)$, and similarly

$$(h(a) - 1) + (h(d) - 1) = (h(b) - 1) + (h(c) - 1).$$

If d is a corner, though, $h(d) - 1 = 0$, so this simplifies very nicely! The idea is that each hook walk comes with a kind of weight $\frac{1}{h(u) - 1)}$, and we have a nice additive identity with inverse weights.

So now consider a rectangular grid of length $k + 1$ by $\ell + 1$. Let's say the weights of the last row are $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}, B$, and the weights in the last column are $\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_\ell}, B$.

Proposition 5

So now if we sum the weights of lattice paths from A to B, the sum of weights is

$$\frac{1}{x_1 x_2 \cdots x_k y_1 \cdots y_\ell}.$$

For example, for a 2 by 2 grid, if the weights are $\frac{1}{x_1 + y_1}, \frac{1}{y_1}, \frac{1}{x_1}$, and 1, then the total weights are

$$\frac{1}{x_1 + y_1} \cdot \frac{1}{y_1} + \frac{1}{x_1 + y_1} \cdot \frac{1}{x_1} = \frac{1}{x_1 y_1}.$$

This is an exercise, and we'll show next lecture that this leads to a proof of the Hook Length Formula!

□