18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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If we have questions about the problem set, we can ask. The official office hours are right after this class on Mondays, but we can also schedule other times.

A few bonus problems will be added to make the problem set more interesting.

Last week, we talked about q-binomial coefficients and q-factorials, which are special cases of another quantity:

Definition 1

The ***q*-multinomial coefficients**

$$\begin{bmatrix} n \\ n_1, n_2, \cdots, n_r \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q![n_2]_q! \cdots [n_r]_q!}$$

can be defined for $n = n_1 + \cdots + n_r$ and all $n_i \ge 0$.

Note that $\begin{bmatrix} n \\ 1, 1, \cdots, 1 \end{bmatrix}_q$ is just $[n]_q!$, and $\begin{bmatrix} n \\ r, n-r \end{bmatrix}_q$ is just $\begin{bmatrix} n \\ r \end{bmatrix}_q$.

Definition 2

A **multiset** is like a regular set, but we allow entries to appear multiple times. For example, we can have 1 appear n_1 times, 2 appear n_2 times, and so on: this will be abbreviated as

$$S = \{1^{n_1}, 2^{n_2}, \cdots, r^{n_r}\}.$$

So now let's consider $w = (w_1, \cdots, w_n)$ as a permutation on the multiset S. We define an inversion very similarly:

Definition 3

An inversion in w is a pair of indices (i, j) where $1 \le i < j \le n$ and $w_i > w_j$. We also define $\text{inv}(w)$ to be the number of inversions in w.

Then the main theorem is very similar:

Theorem 4

For any q-multinomial coefficient,

$$\begin{bmatrix} n \\ n_1, \cdots, n_r \end{bmatrix}_q = \sum_w q^{\text{inv}\, w}$$

where the sum is taken over all permutations of $\{1^{n_1}, 2^{n_2}, \cdots, r^{n_r}\}$.

The proof is similarly by induction, and it is an exercise on the problem set! As a corollary, we know that $\begin{bmatrix} n \\ n_1, \cdots, n_r \end{bmatrix}_q$ is a polynomial in q with positive integer coefficients. The degree of this polynomial is the maximum number of inversions, which happens when we write everything in weakly decreasing order: this is just

$$d = \sum_{1 \le a < b \le r} n_a n_b.$$

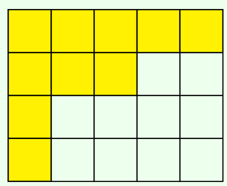
Similarly, we also know that the coefficients are symmetric: $a_i = a_{d-i}$. This follows from the fact that we can just flip the whole sequence around! Basically,

$$\text{inv}(w_1, \cdots, w_n) = d - \text{inv}(w_n, w_{n-1}, \cdots, w_1),$$

since any pair of distinct entries is an inversion in one or the other and d is the total number of pairs of distinct entries. By the way, if our multiset only contains 1s and 2s, so $S = \{1^k 2^{n-k}\}$, there is a correspondence between permutations of S and Young diagrams $\lambda \subseteq k \times (n-k)$.

Example 5

Let $w = (2, 1, 1, 2, 2, 1, 2, 2, 1)$: transform this into a lattice path, going up when we see a 1 and right when we see a 2.



Then the number of squares $|\lambda|$ corresponds to the number of inversions, since we can just match the corresponding 2 and 1!

What if we do $r = 3$? We can think of this as a lattice path in a 3-dimensional box, and we go up, right, or into the page each time we see a $1, 2, 3$ respectively. It's not quite as clean, though.

Let's move on to a new idea!

Let $[n]$ be the set $\{1, 2, \cdots, n\}$. Given any permutation w, we can think of it as a bijective map

$$w : [n] \to [n].$$

We can multiply such maps or take compositions: that's how we multiply permutations! These permutations form a group S_n, called the **symmetric group**.

Fact 6

Stanley's book uses $\mathfrak{S}_n$ instead of S_n.

There's several different ways we can notate permutations:

name	notation	example
1-line notation	$(w_1, \cdots, w_n)$	(2, 5, 7, 3, 1, 6, 8, 4)
2-line notation	$\begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix}$
Cycle notation	$(a_1 a_2 a_3) \cdots$	(125)(3784)(6)

Cycle notation is the most important here: we keep following the permutation until we get back to a point we've already been at. Trivial cycles like (6) are sometimes omitted, and they're called **fixed points** of w.

There's two more: in graphical notation, draw arrows from numbers to where they go. This forms closed polygons.

Finally, we have **matrix notation** (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & j = w(i) \\ 0 & \text{otherwise} \end{cases}$$

which is an $n \times n$ matrix. Here, the matrix is "either this one or the transpose:"

$$w = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We want the one where multiplication works with permutations.

Fact 7

Exercise: is this one the correct one?

Notice that this corresponds to **rook placement** on a chessboard! Place rooks where there are 1s, so there are no rooks attacking each other. There are many problems about non-attacking rook placements, and we'll talk about them later in this class.

What we're going to discuss next is **statistics on permutations**! Basically, we'll somehow map

$$A : S_n \to \{0, 1, 2, \cdots, \}$$

and form a generating function

$$F_A(x) = \sum_{w \in S_n} x^{A(w)}.$$

Definition 8

Two statistics A and B are **equidistributed** if they have the same generating function.

Here are some common statistics that are studied:

- Number of inversions $\text{inv}(w)$
- The length of a permutation $\ell(w)$, defined to be the minimum number ℓ of **adjacent transpositions** (of the form s_i, switching i and $i+1$ but not 1 and n) needed to express w.

It's a fact that we can write any permutation as a sum of adjacent transpositions: just induct on n by switching n into the last spot.

Example 9

We can switch $123 \to 213 \to 231 \to 321$, so $\ell(321)$ is at most 3. It is in fact 3, and this is also the number of inversions.

This is not a coincidence!

Theorem 10

For any permutation w, the length of w is also the number of inversions of w.

So those two statistics aren't just equidistributed: they're actually the same statistic. Let's go back quickly to the generating function and do the proof more carefully:

Theorem 11

The generating function

$$\sum_{w \in S_n} q^{\text{inv } w} = (1+q)(1+q+q^2)+\cdots+(1+q+\cdots+q^{n-1}) = [n]_q!.$$

Proof. This is true by induction on n. This holds for $n=1$, and now let's say it holds for $n-1$.

There are n permutations that can be created by extending an element of S_{n-1}: just put the n somewhere inside. Those n insertions add $n-1, n-2, \cdots, 1, 0$ inversions respectively, so this is

$$\sum_{w \in S_n} q^{\text{inv}(w)} = \sum_{u \in S_{n-1}} q^{\text{inv}(u)}(1+q+q^2+\cdots+q^{n-1} = [n-1]_q![n]_q = [n]_q!,$$

as desired. □

Back to statistics:

- The number of cycles in w, denoted $\text{cyc}(w)$, including fixed points.

For example, $w = (2,5,7,3,1,6,8,4)$ in cycle notation is $(125)(3784)(6)$, so $\text{cyc}(w) = 3$. Note that by degree arguments, this can't be equidistributed with the number of inversions!

Theorem 12

For any n,

$$\sum_{w \in S_n} x^{\text{cyc}(w)} = x(1+x)(2+x)\cdots(n-1+x).$$

Proof. Let's do this by induction. Write our permutations in cycle notation, and let's say we insert n into our permutation. It can either be inserted into one of the existing permutations, or it can go by itself.

There are $n-1$ ways to insert into an existing spot and keep the number of cycles the same, since it does matter where we insert n into an existing cycle, and 1 way to add a new cycle. That's exactly the $((n-1)+x)$ that we want! □