Prof. Yufei Zhao

1. Verify the following asymptotic calculations used in Ramsey number lower bounds:
   (a) For each k, the largest n satisfying $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ has $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$.
   (b) For each k, the maximum value of $n - \binom{n}{k}2^{1-\binom{k}{2}}$ as n ranges over positive integers is $\left(\frac{1}{e} + o(1)\right) k2^{k/2}$.
   (c) For each k, the largest n satisfying $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) 2^{1-\binom{k}{2}} < 1$ satisfies $n = \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{k/2}$.
2. Prove that, if there is a real $p \in [0,1]$ such that
$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$$
then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this show that
$$R(4,t) \geq c\left(\frac{t}{\log t}\right)^{3/2}$$
for some constant $c > 0$.
3. (Extension of Sperner's theorem) Let $\mathcal{F}$ be a collection of subset of $[n]$ that does not contain $k+1$ elements forming a chain: $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Prove that $\mathcal{F}$ is no larger than taking the union of the k levels of the boolean lattice closest to the middle layer.
4. ps1 Let $A_1, \ldots, A_m$ be r-element sets and $B_1, \ldots, B_m$ be s-element sets. Suppose $A_i \cap B_i = \emptyset$ for each i, and for each $i \neq j$, either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$.
5. ps1 Prove that for every positive integer r, there exists an integer K such that the following holds. Let S be a set of rk points evenly spaced on a circle. If we partition $S = S_1 \cup \cdots \cup S_r$ so that $|S_i| = k$ for each i, then, provided $k \geq K$, there exist r congruent triangles where the vertices of the i-th triangle lie in S_i, for each $1 \leq i \leq r$.
6. ps1 Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.
7. ps1⋆ Prove that $[n]^d$ cannot be partitioned into fewer than 2^d sets each of the form $A_1 \times \cdots \times A_d$ where $A_i \subsetneq [n]$.
8. Let $k \geq 4$ and H a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.
9. ps1 Let G be a graph on $n \geq 10$ vertices. Suppose that adding any new edge to G would create a new clique on 10 vertices. Prove that G has at least $8n - 36$ edges.
   (Hint in white:)
10. Prove that there is an absolute constant $c > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$. (A subsequence does not need to be selected from consecutive terms. For example, $(1,2,3)$ is an increasing subsequence of $(1,5,2,4,3)$.)
11. Let G be a graph with n vertices and m edges. Prove that K_n can be written as a union of $O(n^2(\log n)/m)$ copies of G (not necessarily edge-disjoint).
12. ps1 Given a set $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its *projection* onto A). Prove that for every n and k, there exists a set $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every k-element subset A of $[n]$, $\mathcal{F}|_A$ contains all 2^k subsets of A.
13. Let A be a subset of the unit sphere in $\mathbb{R}^3$ (centered at the origin) containing no pair of orthogonal points.

ps1 (a) Prove that A occupies at most 1/3 of the sphere in terms of surface area.

ps1⋆ (b) Prove an upper bound smaller than 1/3 (give your best bound).

14. Let $\boldsymbol{r} = (r_1, \ldots, r_k)$ be a vector of nonzero integers whose sum is nonzero. Prove that there exists a real $c > 0$ (depending on $\boldsymbol{r}$ only) such that the following holds: for every finite set A of nonzero *reals*, there exists a subset $B \subseteq A$ with $|B| \geq c|A|$ such that there do not exist $b_1, \ldots, b_k \in B$ with $r_1 b_1 + \cdots + r_k b_k = 0$.

15. ps1 Prove that every set A of n nonzero integers contains two disjoint subsets B_1 and B_2, such that both B_1 and B_2 are sum-free, and $|B_1| + |B_2| > 2n/3$. Can you do it if A is a set of nonzero reals?

16. ps1⋆ Prove that every graph with n vertices and $m \geq n^{3/2}$ edges contains a pair of vertex-disjoint and isomorphic subgraphs (not necessarily induced) each with at least $cm^{2/3}$ edges, where $c > 0$ is a constant.

17. Let $M(n)$ denote the maximum number of edges in a 3-uniform hypergraph on n vertices without a clique on 4 vertices.
    (a) Prove that $M(n+1)/\binom{n+1}{3} \leq M(n)/\binom{n}{3}$ for all n, and conclude that $M(n)/\binom{n}{3}$ approaches some limit α as $n \to \infty$.
    (This limit is called the *Turán density* of the hypergraph $K_4^{(3)}$, and its exact value is currently unknown and is a major open problem.)
    (b) Prove that for every $\delta > 0$, there exists $\epsilon > 0$ and n_0 so that every 3-uniform hypergraph with $n \geq n_0$ vertices and at least $(\alpha + \delta)\binom{n}{3}$ edges must contain at least $\epsilon\binom{n}{4}$ copies of the clique on 4 vertices.

18. Using the alteration method, prove that the Ramsey number $R(4, k)$ satisfies $R(4, k) \geq c(k/\log k)^2$ for some constant $c > 0$.

19. Prove that every 3-uniform hypergraph with n vertices and $m \geq n$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least $cn^{3/2}/\sqrt{m}$, where $c > 0$ is a constant.

20. (Zarankiewicz problem) Prove that for every positive integer $k \geq 2$, there exists a constant $c > 0$ such that for every n, there exists an $n \times n$ matrix with $\{0, 1\}$ entries, with at least $cn^{2-2/(k+1)}$ 1's, such that there is no $k \times k$ submatrix consisting of all 1's.

21. ps2 Fix k. Prove that there exists a constant $c_k > 1$ so that for every sufficiently large n, there exists a collection $\mathcal{F}$ of at least c_k^n subsets of $[n]$ such that for every k distinct $F_1, \ldots, F_k \in \mathcal{F}$, all 2^k intersections $\bigcap_{i=1}^k G_i$ are nonempty, where each G_i is either F_i or $[n] \setminus F_i$.

22. *Acute sets in* $\mathbb{R}^n$
    (a) Prove that there exists a family of $\Omega((2/\sqrt{3})^n)$ subsets of $[n]$ containing no three distinct members A, B, C satisfying $A \cap B \subseteq C \subseteq A \cup B$.
    (b) Prove that there exists a set of $\Omega((2/\sqrt{3})^n)$ points in $\mathbb{R}^n$ so that all angles determined by three points from the set are acute.
    Remark: The current best lower and upper bounds for the maximum size of an "acute set" in $\mathbb{R}^n$ (i.e., spanning only acute angles) are $2^{n-1} + 1$ and $2^n - 1$ respectively.
    ps2 (c) Prove that there exists a constant $c > 1$ such that for every n, there are at least c^n points in $\mathbb{R}^n$ so that the angle spanned by every three distinct points is at most 61°.

ps2⋆ 23. *Covering complements of sparse graphs by cliques*

(a) Prove that every graph with n vertices and minimum degree $n-d$ can be written as a union of $O(d^2 \log n)$ cliques.

(b) Prove that every bipartite graph with n vertices on each side of the vertex bipartition and minimum degree $n-d$ can be written as a union of $O(d \log n)$ complete bipartite graphs (assume $d \geq 1$).

ps2⋆ 24. Let $G=(V,E)$ be a graph with n vertices and minimum degree $\delta \geq 2$. Prove that there exists $A \subseteq V$ with $|A| \leq Cn(\log \delta)/\delta$, where $C>0$ is a constant, so that every vertex in $V \setminus A$ contains at least one neighbor in A and at least one neighbor not in A.

25. Let X be a nonnegative real-valued random variable. Suppose $\mathbb{P}(X=0)<1$. Prove that

$$\mathbb{P}(X=0) \leq \frac{\operatorname{Var} X}{\mathbb{E}[X^2]}.$$

ps2 26. Let X be a random variable with mean μ and variance σ^2. Prove that for all $\lambda>0$,

$$\mathbb{P}(X \geq \mu+\lambda) \leq \frac{\sigma^2}{\sigma^2+\lambda^2}.$$

27. What is the threshold function for $G(n,p)$ to contain a cycle?

ps2 28. Show that, for each fixed k, there is a sequence p_n such that

$$\mathbb{P}(G(n,p_n) \text{ has a connected component with exactly } k \text{ vertices}) \to 1 \quad \text{as } n \to \infty.$$

29. Let $p=(\log n+f(n))/n$. Show that, as $n \to \infty$,

$$\mathbb{P}(G(n,p) \text{ has no isolated vertices}) \to \begin{cases} 0 & \text{if } f(n) \to -\infty, \\ 1 & \text{if } f(n) \to \infty. \end{cases}$$

ps2 30. Let $v_1=(x_1,y_1),\dots,v_n=(x_n,y_n) \in \mathbb{Z}^2$ with $|x_i|,|y_i| \leq 2^{n/2}/(100\sqrt{n})$ for all $i \in [n]$. Show that there are two disjoint sets $I,J \subseteq [n]$, not both empty, such that $\sum_{i\in I} v_i = \sum_{j \in J} v_j$.

ps2⋆ 31. Prove that there is an absolute constant $c>0$ so that the following holds. For every prime p and every $A \subseteq \mathbb{Z}/p\mathbb{Z}$ with $|A|=k$, there exists an integer x so that $\{xa : a \in A\}$ intersects every interval of length at least $cp/\sqrt{k}$ in $\mathbb{Z}/p\mathbb{Z}$.

ps2⋆ 32. Let $S_1,\dots,S_k$ be subsets of $[n]$. Prove that if $k \leq 1.99n/\log_2 n$ and n is sufficiently large, then there are two distinct subsets $X,Y \subseteq [n]$ such that $|X \cap S_i| = |Y \cap S_i|$ for all $i \in [k]$.

In addition, show that there is some constant C such that the claim is false for $k \geq Cn/\log_2 n$. What is the best constant C?

ps2⋆ 33. Let X be a collection of pairwise orthogonal unit vectors in $\mathbb{R}^n$ and suppose that the projection of each of these vectors on the first k coordinates has norm at least ϵ. Show that $|X| \leq k/\epsilon^2$, and show that this is tight if $\epsilon^2 = k/2^r < 1$ for some integer r.

ps2⋆ 34. Prove that there is a constant $c>0$ so that every hyperplane containing the origin in $\mathbb{R}^n$ intersects at least c-fraction of the 2^n closed unit balls centered at $\{-1,1\}^n$.
(Give your best c. Can you get $c \geq 3/8$? It is conjectured that $c=1/2$ works.)

ps2 35. Prove that, with probability approaching 1 as $n \to \infty$, $G(n, n^{-1/2})$ has at least $cn^{3/2}$ edge-disjoint triangles, where $c > 0$ is some constant.
(Hint in white:)

ps2 36. *Simple nibble.* Prove that for some constant C, with probability approaching 1 as $n \to \infty$,
(a) $G(n, Cn^{-2/3})$ has at least $n/100$ vertex-disjoint triangles.
(b) $G(n, Cn^{-2/3})$ has at least $0.33n$ vertex-disjoint triangles
(Hint in white:)
(You are asked to solve the above problem using the second moment method. Later in the course we will learn a different method to solve this problem.)

37. Let $X \sim \text{Binomial}(n, p)$. Prove that for $0 < q \le p < 1$,

$$\mathbb{P}(X \le nq) \le e^{-nH(q\|p)} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X \le nq) = -H(\|p)$$

and for $0 < p \le q < 1$,

$$\mathbb{P}(X \ge nq) \le e^{-nH(q\|p)} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X \ge nq) = -H(\|p),$$

where

$$H(q\|p) := q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p}.$$

is known as the *relative entropy* or *Kullback-Leibler divergence*, in this case, between two Bernoulli distributions.

38. Prove that there is a constant $C > 0$ so that, with probability $1 - o(1)$ as $n \to \infty$, the maximum number of edges in a bipartite subgraph of $G(n, 1/2)$ is at most $n^2/8 + Cn^{3/2}$.

39. (a) Prove that there is some constant $c > 1$ so that there exists $S \subset \{0,1\}^n$ with $|S| \ge c^n$ so that every pair of points in S differ in at least $n/4$ coordinates.
(b) Prove that there is some constant $c > 1$ so that the the unit sphere in $\mathbb{R}^n$ contains at least c^n points, where each pair of points is at distance at least 1 apart.

ps3 40. *Planted clique.* Give a deterministic polynomial-time algorithm solving the following problem so that it succeeds over the random input with probability approaching 1 as $n \to \infty$:
Input: an n-vertex unlabeled graph G created as the union of $G(n, 1/2)$ and a clique on vertex subset of size $t = \lfloor 100\sqrt{n \log n} \rfloor$
Output: a clique in G of size t

ps3 41. Show that it is possible to color the edges of K_n with at most $3\sqrt{n}$ colors so that there are no monochromatic triangles.

42. Prove that there is some constant C so that it is possible to color the vertices of every k-uniform k-regular hypergraph using at most $k/\log k$ colors so that every edge has at most $C \log k$ vertices of each color.

ps3 43. Prove that there is some constant $c > 0$ so that given a graph and a set of k acceptable colors for each vertex such that every color is acceptable for at most ck neighbors of each vertex, there is always a proper coloring where every vertex is assigned one of its acceptable colors.

ps3⋆ 44. Prove that there is a constant $C > 0$ so that for every sufficiently small $\epsilon > 0$, one can choose exactly one point inside each grid square $[n, n+1) \times [m, m+1) \subset \mathbb{R}^2$, $m, n \in \mathbb{Z}$, so that every rectangle of dimensions ϵ by $(C/\epsilon)\log(1/\epsilon)$ in the plane (not necessarily axis-aligned) contains at least one chosen point.

ps3 45. Prove that, for every $\epsilon > 0$, there exists ℓ_0 and some $(a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}}$ such that for every $\ell > \ell_0$ and every $i > 1$, the vectors $(a_i, a_{i+1}, \dots, a_{i+\ell-1})$ and $(a_{i+\ell}, a_{i+\ell+1}, \dots, a_{i+2\ell-1})$ differ in at least $(\frac{1}{2} - \epsilon)\ell$ coordinates.

ps3 46. A *periodic path* in a graph G with respect to a vertex coloring $f\colon V(G) \to [k]$ is a path $v_1 v_2 \dots v_{2\ell}$ for some positive integer ℓ with $f(v_i) = f(v_{i+\ell})$ for each $i \in [\ell]$ (reminder: no repeated vertices allowed in a path).

Prove that for every Δ, there exists k so that every graph with maximum degree at most Δ has a vertex-coloring using k colors with no periodic paths.

ps3 47. Prove that every graph with maximum degree Δ can be properly edge-colored using $O(\Delta)$ colors so that every cycle contains at least three colors.

(A *proper edge-coloring* is one where no two adjacent edges receive the same color.)

ps3⋆ 48. Prove that for every Δ, there exists g so that every bipartite graph with maximum degree Δ and girth at least g can be properly edge-colored using $\Delta + 1$ colors so that every cycle contains at least three colors.

ps3⋆ 49. Prove that for every positive integer r, there exists C_r so that every graph with maximum degree Δ has a *proper* vertex coloring using at most $C_r \Delta^{1+1/r}$ colors so that every vertex has at most r neighbors of each color.

50. Let $H = (V, E)$ be a hypergraph satisfying, for some $\lambda > 1/2$,

$$\sum_{f \in E : v \in f} \lambda^{|f|} \le \frac{1}{2} - \frac{1}{4\lambda} \quad \text{for every } v \in V$$

(here $|f|$ is then number of vertices in the edge f). Prove that H is 2-colorable.

51. Prove that there exists k_0 and a red/blue coloring of $\mathbb{Z}$ without any monochromatic k-term arithmetic progressions with $k \ge k_0$ and common difference less than 1.99^k.

52. *Vertex-disjoint cycles in digraphs.* (Recall that a directed graph is *k-regular* if all vertices have in-degree and out-degree both equal to k. Also, cycles cannot repeat vertices.)

ps3 (a) Prove that every k-regular directed graph has at least $ck/\log k$ vertex-disjoint directed cycles, where $c > 0$ is some constant.

ps3⋆ (b) Prove that every k-regular directed graph has at least ck vertex-disjoint directed cycles, where $c > 0$ is some constant.

(Hint in white:)

ps3⋆ 53. Prove that there is a constant $c > 0$ so that every $n \times n$ matrix where no entry appears more than cn times contains cn disjoint Latin transversals.

(Hint in white:)

54. (a) *Generalization of Cayley's formula.* Using Prüfer codes, prove the identity

$$x_1 x_2 \cdots x_n (x_1 + \cdots + x_n)^{n-2} = \sum_T x_1^{d_T(1)} x_2^{d_T(2)} \cdots x_n^{d_T(n)}$$

where the sum is over all trees T on n vertices labeled by $[n]$ and $d_T(i)$ is the degree of vertex i in T.

(b) *Independence property for uniform spanning tree of* K_n. Let F be a forest with vertex set $[n]$, with components having $f_1, \dots, f_s$ vertices so that $f_1 + \cdots + f_s = n$. Prove that the number of trees on the vertex set $[n]$ that contain F is exactly $n^{n-2} \prod_{i=1}^{s} (f_i / n^{f_i - 1})$. Deduce that if H_1 and H_2 are vertex-disjoint subgraphs of K_n, then for a uniformly random spanning tree T of K_n, the events $H_1 \subseteq T$ and $H_2 \subseteq T$ are independent.

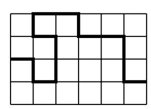
ps3⋆ (c) *Packing rainbow spanning trees.* Prove that there is a constant $c > 0$ so that for every edge-coloring of K_n where each color appears at most cn times, there exist at least cn edge-disjoint spanning trees, where each spanning tree has all its edges colored differently.

ps4 55. Let $G = (V, E)$ be a graph. Color every edge with red or blue independently and uniformly at random. Let E_0 be the set of red edges and E_1 the set of blue edges. Let $G_i = (V, E_i)$ for each $i = 0, 1$. Prove or disprove:

$$\mathbb{P}(G_0 \text{ and } G_1 \text{ are both connected}) \leq \mathbb{P}(G_0 \text{ is connected})^2.$$

ps4 56. A set family $\mathcal{F}$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ each be a collection of subsets of $[n]$ and suppose that each $\mathcal{F}_i$ is intersecting. Prove that $\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| \leq 2^n - 2^{n-k}$.

57. Let $G_{m,n}$ be the grid graph on vertex set $[m] \times [n]$ (m vertices wide and n vertices tall). A *horizontal crossing* is a path that connects some left-most vertex to some right-most vertex. See below for an example of a horizontal crossing in $G_{7,5}$.



Let $H_{m,n}$ denote the random subgraph of $G_{m,n}$ obtained by keeping every edge with probability $1/2$ independently.

Let $\mathsf{RSW}(k)$ denote the following statement: there exists a constant $c_k > 0$ such that for all positive integers n, $\mathbb{P}(H_{kn,n} \text{ has a horizontal crossing}) \geq c_k$.

ps4 (a) Prove that $\mathsf{RSW}(2)$ implies $\mathsf{RSW}(100)$.

ps4⋆ (b) Prove $\mathsf{RSW}(1)$.

(c) (Challenging. Not to be turned in) Prove $\mathsf{RSW}(2)$.

ps4 58. Let U_1 and U_2 be increasing events and D a decreasing event of independent boolean random variables. Suppose U_1 and U_2 are independent. Prove that $\mathbb{P}(U_1 | U_2 \cap D) \leq \mathbb{P}(U_1 | U_2)$.

ps4 59. *Coupon collector.* Let $s_1, \dots, s_m$ be independent random elements in $[n]$ (not necessarily uniform or identically distributed; chosen with replacement) and $S = \{s_1, \dots, s_m\}$. Let I and J be disjoint subsets of $[n]$. Prove that $\mathbb{P}(I \cup J \subseteq S) \leq \mathbb{P}(I \subseteq S)\mathbb{P}(J \subseteq S)$.
(Hint in white:)

ps4⋆ 60. Prove that there exist $c, \epsilon > 0$ such that if $A_1, \dots, A_k$ are increasing events of independent boolean random variables with $\mathbb{P}(A_i) \leq \epsilon$ for all i, then, letting X denote the number of events A_i that occur, one has $\mathbb{P}(X = 1) \leq 1 - c$. (Give your largest c.)

ps4 61. Prove that with probability $1 - o(1)$, the size of the largest subset of vertices of $G(n, 1/2)$ inducing a triangle-free subgraph is $\Theta(\log n)$.

62. *Lower tails of small subgraph counts.* Fix graph H and $\epsilon \in (0,1]$. Let X_H denote the number of copies of H in $G(n,p)$. Prove that for all n and $0 < p < 1/2$,

$$\mathbb{P}(X_H \le (1-\epsilon)\mathbb{E}X_H) = e^{-\Theta_{H,\epsilon}(\Phi_H)} \quad \text{where } \Phi_H := \min_{H' \subseteq H : e(H') > 0} n^{v(H')} p^{e(H')}.$$

Here the hidden constants in $\Theta_{H,\epsilon}$ may depend on H and ϵ (but not on n and p).

63. *Vertex-disjoint triangles in $G(n,p)$ again.* Using Janson inequalities this time, give another solution to Problem 36 in the following generality.

ps4 (a) Prove that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that such that with probability $1 - o(1)$, $G(n, C_\epsilon n^{-2/3})$ contains at least $(1/3 - \epsilon)n$ vertex-disjoint triangles.

ps4⋆ (b) Compare the the dependence of the optimal C_ϵ on ϵ you obtain using the method in Problem 36 versus this problem (don't worry about leading constant factors).

ps4⋆ 64. Show that $\operatorname{ch}(G(n,1/2)) = (1+o(1))\frac{n}{2\log_2 n}$ with probability $1 - o(1)$.
Here $\operatorname{ch}(G)$ is the *list-chromatic number* (also called *choosability*) of a graph G and it is defined to the minimum k such that if every vertex of G is assigned a list of k acceptable colors, then there exists a proper coloring of G where every vertex is colored by one of its acceptable colors.

ps4 65. For each part, prove that there is some constant $c > 0$ so that, for all $\lambda > 0$,

$$\mathbb{P}(|X - \mathbb{E}X| \ge \lambda\sqrt{\operatorname{Var} X}) \le 2e^{-c\lambda^2}.$$

(Such families of random variables are called *sub-Gaussian.*)

(a) X is the number of triangles in $G(n,1/2)$.

(b) X is the number of inversions of a uniform random permutation of $[n]$ (an *inversion* of $\sigma \in S_n$ is a pair (i,j) with $i < j$ and $\sigma(i) > \sigma(j)$).

ps4⋆ 66. Let $k \le n/2$ be positive integers and G an n-vertex graph with average degree at most n/k. Prove that a uniform random k-element subset of the vertices of G contains an independent set of size at least ck with probability at least $1 - e^{-ck}$, where $c > 0$ is a constant.

ps5 67. True or False: In the definition of a martingale, the condition $\mathbb{E}[X_n | X_{n-1} = x_{n-1}, \dots, X_0 = x_0] = x_{n-1}$ may be replaced by simply $\mathbb{E}[X_n | X_{n-1} = x_{n-1}] = x_{n-1}$.

ps5 68. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ and n_0 such that for all $n \ge n_0$ and $S_1, \dots, S_m \subset [2n]$ with $m \le 2^{\delta n}$ and $|S_i| = n$ for all $i \in [m]$, there exists a function $f\colon [2n] \to [n]$ so that $(1 - e^{-1} - \epsilon)n \le |f(S_i)| \le (1 - e^{-1} + \epsilon)n$ for all $i \in [m]$.

ps5 69. *Simultaneous bisections.* Fix Δ. Let $G_1, \dots, G_m$ with $m = 2^{o(n)}$ be connected graphs of maximum degree at most Δ on the same vertex set V with $|V| = n$. Prove that there exists a partition $V = A \cup B$ so that every G_i has $(1+o(1))e(G_i)/2$ edges between A and B.

ps5 70. Show that for every $\epsilon > 0$ there exists $C > 0$ so that every $S \subset [4]^n$ with $|S| \ge \epsilon 4^n$ contains four elements whose pairwise Hamming distance at least $n - C\sqrt{n}$.

ps5⋆ 71. *Tighter concentration of chromatic number*

(a) Prove that with probability $1 - o(1)$, every vertex subset of $G(n,1/2)$ with at least $n^{1/3}$ vertices contains an independent set of size at least $c \log n$, where $c > 0$ is some constant.

(b) Prove that there exists some function $f(n)$ and constant C such that for all $n \ge 2$,

$$\mathbb{P}(f(n) \le \chi(G(n,1/2)) \le f(n) + C\sqrt{n}/\log n) \ge 0.99.$$

ps5⋆ 72. Let $G = (V, E)$ with chromatic number $\chi(G) = k$ and S a uniform random subset of V. Prove that for every $t \geq 0$,

$$\mathbb{P}(\chi(G[S]) \leq k/2 - t) \leq e^{-ct^2/k},$$

where $c > 0$ is a constant and $G[S]$ is the subgraph induced by S.

ps5⋆ 73. Prove that for all n there exists some $k \sim 2\log_2 n$ and some n-vertex graph that contains every graph on k vertices as an induced subgraph.

ps5⋆ 74. Prove that there exists a constant $c > 0$ so that the following holds. Let G be a d-regular graph and $v_0 \in V(G)$. Let $m \in \mathbb{N}$ and consider a simple random walk $v_0, v_1, \ldots, v_m$ where each v_{i+1} is a uniform random neighbor of v_i. For each $v \in V(G)$, let X_v be the number times that v appears among $v_0, \ldots, v_m$. For that for every $v \in V(G)$ and $\lambda > 0$

$$\mathbb{P}\left(X_v - \frac{1}{d} \sum_{w \in N(v)} X_w \geq \lambda + 1 \right) \leq 2e^{-c\lambda^2/m}$$

Here $N(v)$ is the neighborhood of v.

ps5⋆ 75. Let $\mathsf{maxcut}(G)$ denote the maximum number of edges in a bipartite subgraph of G. Prove there is a constant $c > 0$ so that $\mathsf{maxcut}(G(n, 1/2)) > n^2/8 + cn^{3/2}$ with probability $1 - o(1)$.

For the next three exercises, use Talagrand's inequality

ps5 76. Let Q be a subset of the unit sphere in $\mathbb{R}^n$. Let $\boldsymbol{x} \in [0,1]^n$ be a random vector with independent coordinates. Let $X = \sup_{\boldsymbol{q} \in Q} \langle \boldsymbol{x}, \boldsymbol{q} \rangle$ and m a median of X. Let $t > 0$. Prove

$$\mathbb{P}(|X - m| \geq t) \leq 4e^{-t^2/4}.$$

ps5⋆ 77. Prove that there are constants $c, C > 0$ such that if A is a symmetric $n \times n$ matrix with independent entries in $[-1, 1]$, then the second largest eigenvalue $\lambda_2(A)$ satisfies

$$\mathbb{P}(|\lambda_2(A) - \mathbb{E}\lambda_2(A)| > t) \leq Ce^{-ct^2}.$$

(Hint: use this Courant–Fischer characterization of $\lambda_2(X)$: for every pair of unit vectors $u, v \in \mathbb{R}^n$, there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $w = au + bv$ satisfying $w^t X w \leq \lambda_2(X)$.)

78. Let $q = q_n \gg n$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two random sequences whose entries are chosen independently and uniformly at random from $[q]$. Let X be the length of the longest common subsequence between $\boldsymbol{x}$ and $\boldsymbol{y}$ (i.e., X is the largest k such that there exist $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ with $x_{i_1} = y_{j_1}, \ldots, x_{i_k} = y_{j_k}$). Show that with probability $1 - o(1)$, X lies within $\sqrt{n}$ of its median.

Entropy methods (You are encouraged to find solutions using entropy)

79. (Submodularity) Prove that $H(X, Y, Z) + H(X) \leq H(X, Y) + H(X, Z)$.

ps5⋆ 80. (Uniquely decodable codes) Let $[r]^*$ denote the set of all finite strings of elements in $[r]$. Let A be a finite subset of $[r]^*$ and suppose no two distinct concatenations of sequences in A can produce the same string. Prove that $\sum_{a \in A} r^{-|a|} \leq 1$ where $|a|$ is the length of $a \in A$.

ps5 81. Let $\mathcal{G}$ be a family of graphs on vertices labeled by $[2n]$ such that the intersection of every pair of graphs in $\mathcal{G}$ contains a perfect matching. Prove that $|\mathcal{G}| \leq 2^{\binom{2n}{2} - n}$.

ps5⋆ 82. Let X, Y, Z be independent $\mathbb{Z}$-valued random variables. Prove that

$$2H(X+Y+Z) \le H(X+Y) + H(X+Z) + H(Y+Z).$$

ps5⋆ 83. *Triangles versus vees in a directed graph.* Let V be a finite set, $E \subseteq V \times V$, and

$$\triangle = \left|\left\{(x,y,z) \in V^3 : (x,y),(y,z),(z,x) \in E\right\}\right|$$

(i.e., cyclic triangles; note the direction of edges) and

$$\wedge = \left|\left\{(x,y,z) \in V^3 : (x,y),(x,z) \in E\right\}\right|.$$

Prove that $\triangle \le \wedge$.