10. Quantum mechanics for fermions

10.1. **Feynman calculus in the supercase.** Wick's theorem allows us to extend Feynman calculus to the supercase. Namely, let $V = V_0 \oplus V_1$ be a finite dimensional real superspace with a supervolume element dv, equipped with a symmetric nondegenerate form $B = B_0 \oplus B_1$ ($B_0 > 0$). Let $S(v) = \frac{1}{2}B(v,v) + \sum_{r\geq 3} \frac{B_r(v,v,\ldots,v)}{r!}$ be an even function on V (the action). Note that B_r, $r \geq 3$ can contain mixed terms involving both odd and even variables, e.g. $x\xi_1\xi_2$ (the so called "Yukawa term"). We will consider the integral

$$I(\hbar) = \int_V \ell_1(v_0)\cdots\ell_n(v_0)\lambda_1(v_1)\cdots\lambda_p(v_1)e^{-S(v)/\hbar}dv.$$

(where v_0, v_1 are the even and odd components of v). Then this integral has an expansion in $\hbar$ written in terms of Feynman diagrams. Since v has both odd and even part, these diagrams will contain "odd" and "even" edges (which could be depicted by straight and wiggly lines respectively). More precisely, let us write

$$B_r(v,v,\ldots,v) = \sum_{s=0}^{r} \binom{r}{s} B_{s,r-s}(v_1,\ldots,v_1,v_0,\ldots,v_0),$$

where $B_{s,r-s}$ has homogeneity degree s with respect to v_1 and $r-s$ with respect to v_0 (i.e. it will be nonzero only for even s). Then to each term $B_{s,r-s}$ we assign an $(s,r-s)$-valent flower, i.e. a flower with s odd and $r-s$ even outgoing edges, and for the set of odd outgoing edges, it has been specified which orderings are even. Then, given an arrangement of flowers, for every pairing σ of outgoing edges, we can define an amplitude F_σ by contracting the tensors $-B_{s,r-s}$ (and being careful with the signs). It is easy to check that all pairings giving the same graph will contribute to $I(\hbar)$ with the same sign, and this we have almost the same formula as in the bosonic case:

$$I(\hbar) = (2\pi)^{\dim V_0/2}\hbar^{\frac{\dim V_0 - \dim V_1}{2}} \frac{\mathrm{Pf}(-B_1)}{\sqrt{\det B_0}} \sum_\Gamma \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F_\Gamma,$$

where the summation is taken over graphs with n even and p odd outgoing edges.

Remark. More precisely, we can define the sign ε_σ of a pairing σ as follows: label outgoing edges by $1,\ldots,$ starting from the first flower, then second, etc., so that the labeling is even on each flower. Then write the labels in a sequence, enumerating (in any order) the pairs defined by σ (the element with the smaller of the two labels goes first). The sign ε_σ is by definition the sign of this ordering (as a permutation of $1,\ldots$). Then F_Γ is F_σ for any pairing σ yielding Γ which is *positive*, i.e. such that $\varepsilon_\sigma = 1$. For negative pairing, $F_\Gamma = -F_\sigma$.

In most (but not all) situations considered in physics, the action is quadratic in the fermionic variables, i.e. $S(v) = S_b(v_0) - S_f(v_0)(v_1, v_1)$, where $S_f(v_0)$ is a skew-symmetric bilinear form on ΠV_1. In this case, using fermionic Wick's theorem, we can perform exact integration with respect to v_1, and reduce $I(\hbar)$ to a purely bosonic integral. For example, if we have only ℓ_i and no λ_i, we have

$$I(\hbar) = \int_{V_0} \ell_1(v_0)\cdots\ell_n(v_0)e^{-S_b(v_0)/\hbar}\mathrm{Pf}(S_f(v_0))dv_0.$$

In this situation, all vertices which have odd outgoing edges, will have only two of them, and therefore in any Feynman diagram with even outgoing edges, odd lines form nonintersecting simple curves, called fermionic loops (in fact, the last formula is nothing but the result of regarding these loops as a new kind of vertices – convince yourself of this). In this case, there is the following simple way of assigning signs to Feynman diagrams. For each vertex with two odd outgoing edges, we orient the first edge inward and the second one outward. We allow only connections (pairings) that preserve orientations (so the fermionic loops become oriented). Then the sign is $(-1)^r$, where r is the number of fermionic loops (i.e. each fermionic loop contributes a minus sign).

10.2. **Fermionic quantum mechanics.** Let us now pass from finite dimensional fermionic integrals to quantum mechanics, i.e. integrals over fermionic functions of one (even) real variable t.

Let us first discuss fermionic classical mechanics, in the Lagrangian setting. Its difference with the bosonic case is that the "trajectory" of the particle is described by an *odd*, rather than even, function of one variable, i.e. $\psi : \mathbb{R} \to \Pi V$, where V is a vector space. Mathematically this means that the space of fields (=trajectories) is an odd vector space $\Pi C^\infty(\mathbb{R}, V)$. A Lagrangian $\mathcal{L}(\psi)$ is a local expression in such a field (i.e. a polynomial in $\psi, \dot\psi, \ldots$), and an action is the integral $S = \int_{\mathbb{R}} \mathcal{L} dt$. This means that the action is an element of the space $\Lambda C_0^\infty(\mathbb{R}, V)^*$.

Consider for example the theory of a single scalar-valued free fermion $\psi(t)$. By definition, the Lagrangian for such a theory is $\mathcal{L} = \frac{1}{2}\psi\dot\psi$, i.e. the action is $S = \frac{1}{2}\int \psi\dot\psi dt$.

This Lagrangian is the odd analog of the Lagrangian of a free particle, $\dot q^2/2$.

Remark. Note that $\psi\dot\psi \neq \frac{d}{dt}(\psi^2/2) = 0$, since $\psi\dot\psi = -\dot\psi\psi$, so this Lagrangian is "reasonable". On the other hand, the same Lagrangian would be unreasonable in the bosonic case, as it would be a total derivative, and hence the action would be zero. Finally, note that it would be equally unreasonable to use in the fermionic case the usual bosonic Lagrangian $\frac{1}{2}(\dot q^2 - m^2 q^2)$; it would identically vanish if q were odd-valued.

The Lagrangian $\mathcal{L}$ is invariant under the group of reparametrizations $\mathrm{Diff}_+(\mathbb{R})$, and the Euler-Lagrange equation for this Lagrangian is $\dot\psi = 0$ (i.e. no dynamics). Theories with such properties are called *topological field theories.*

Let us now turn to quantum theory in the Lagrangian setting, i.e. the theory given by the Feynman integral $\int \psi(t_1)\cdots\psi(t_n)e^{iS(\psi)}D\psi$. In the bosonic case, we "integrated" such expressions over the space $C_0^\infty(\mathbb{R})$. This integration did not make immediate sense because of failure of measure theory in infinite dimensions. So we had to make sense of this integration in terms of $\hbar$-expansion, using Wick's formula and Feynman diagrams. In the fermionic case, the situation is analogous. Namely, now we must integrate functions over $\Pi C_0^\infty(\mathbb{R})$, which are elements of $\Lambda\mathcal{D}(\mathbb{R})$, where $\mathcal{D}(\mathbb{R})$ is the space of distributions on $\mathbb{R}$. Although in the fermionic case we don't need measure theory (as integration is completely algebraic), we still have trouble defining the integral: recall that by definition the integral should the top coefficient of the integrand as the element of $\Lambda\mathcal{D}(\mathbb{R})$, which makes no sense since in the exterior algebra of an infinite dimensional space there is no top component. Thus we have to use the same strategy as in the bosonic case, i.e. Feynman diagrams.

Let us, for instance, define the quantum theory for a free scalar valued fermion, i.e one described by the Lagrangian $\mathcal{L} = \frac{1}{2}\psi\dot\psi$. According to the yoga we used in the bosonic case, the two point function of this theory $< \psi(t_1)\psi(t_2) >$ should be the function $G(t_1 - t_2)$, where G is the solution of the differential equation $-i\frac{dG}{dt} = \delta(t)$.

The general solution of this equation has the form $G(t) = -\frac{1}{2i}\mathrm{sign}(t) + C$. Because of the fermionic nature of the field $\psi(t)$, it is natural to impose the requirement that $G(-t) = -G(t)$, i.e that the correlation functions are antisymmetric; this singles out the solution $G(t) = -\frac{1}{2i}\mathrm{sign}(t)$ (we also see from this condition that we should set $G(0) = 0$). As usual, the $2n$-point correlation functions are defined by the Wick formula. That is, for distinct t_i,

$$< \psi(t_1)\cdots\psi(t_{2n}) >= (2n-1)!!(i/2)^n \mathrm{sign}(\sigma),$$

where σ is the permutation that orders t_i in the decreasing order. If at least two points coincide, the correlation function is zero.

Thus we see that the correlation functions are invariant under $\mathrm{Diff}_+(\mathbb{R})$. In other words, using physical terminology, we have a *topological quantum field theory.*

Note that the correlation functions in the Euclidean setting for this model are the same as in the Minkowski setting, since they are (piecewise) constant in t_i. In particular, they don't decay at infinity, and hence our theory does not have the clustering property.

We have considered the theory of a massless fermionic field. Consider now the massive case. This means, we want to add to the Lagrangian a quadratic term in ψ which does not contain derivatives. If we have only one field ψ, the only choice for such term is ψ^2, which is zero. So in the massive case we must have at least two fields. Let us therefore consider the theory of two fermionic fields ψ_1, ψ_2 with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi_1\dot\psi_1 + \psi_2\dot\psi_2 - m\psi_1\psi_2),$$

where $m > 0$ is a mass parameter. The Green's function for this model satisfies the differential equation

$$G' - MG = i\delta,$$

where $M = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$ (and G is a 2 by 2 matrix valued function). The general solution of this equation that satisfies the antisymmetry condition $G^T(-t) = -G(t)$ (which we will impose as in the massless case) has the form

$$G(t) = -(\frac{1}{2i}\mathrm{sign}(t)I + aM)e^{Mt},$$

where I is the identity matrix, and a is a number. Furthermore, it is natural to require that the theory at hand satisfies the clustering property (after being Wick-rotated). This means, $G(-it)$ decays at infinity for real t. It is easy to compute that this condition is satisfied for only one value of a, namely $a = 1/2m$. For this value of a, the solution has the form $G(t) = \mp iP_\pm e^{-iMt}$ for $\mp t > 0$, where $P_\pm$ is the projector to the eigenspace of M with eigenvalue $\pm im$ (and $G(0) = 0$).

Remark. It is easy to generalize this analysis to the case when ψ takes values in a positive definite inner product space V, and $M : V \to V$ is a skewsymmetric operator, since such a situation is a direct sum of the situations considered above.

In the case when M is nondegenerate, one can define the corresponding theory with interactions, i.e. with higher than quadratic terms in ψ. Namely, one defines the correlators as sums of amplitudes of appropriate Feynman diagrams. We leave it to the reader to work out this definition, by analogy with the finite dimensional case which we have discussed above.

10.3. **Super Hilbert spaces.** The space of states of a quantum system is a Hilbert space. As we plan to do Hamiltonian quantum mechanics for fermions, we must define a superanalog of this notion.

Recall that a sesquilinear form on a complex vector space is a form $(,)$ which is additive in each variable, and satisfies the conditions $(ax, y) = \bar{a}(x, y)$, $(x, ay) = a(x, y)$ for $a \in \mathbb{C}$.

Now suppose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a $\mathbb{Z}/2$-graded complex vector space.

Definition 10.1. (i) A Hermitian form on $\mathcal{H}$ is an even sesquilinear form $(,)$, such that $(x, y) = \overline{(y, x)}$ for even x, y, and $(x, y) = -\overline{(y, x)}$ for odd x, y.

(ii) A Hermitian form is positive definite if $(x, x) > 0$ for even $x \neq 0$, and $-i(x, x) > 0$ for odd $x \neq 0$. A super Hilbert space is a superspace with a positive definite Hermitian form $(,)$, which is complete in the corresponding norm.

(iii) Let $\mathcal{H}$ be a super Hilbert space, and $T : \mathcal{H}_0 \oplus \Pi\mathcal{H}_1 \to \mathcal{H}_0 \oplus \Pi\mathcal{H}_1$ be a linear operator between the underlying purely even spaces. The Hermitian adjoint operator T^* is defined by the equation $(x, Ty) = (-1)^{p(x)p(T)}(T^*x, y)$, where p denotes the parity.

10.4. **The Hamiltonian setting for fermionic quantum mechanics.** Let us now discuss what should be the Hamiltonian picture for the theory of a free fermion. More precisely, let V be a positive definite finite dimensional real inner product space, and consider the Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \dot{\psi}) - (\psi, M\psi))$, where $\psi : \mathbb{R} \to \Pi V$, and $M : V \to V$ is a skew-symmetric operator.

To understand what the Hamiltonian picture should be, let us compare with the bosonic case. Namely, consider the Lagrangian $\mathcal{L}_b = \frac{1}{2}(\dot{q}^2 - m^2q^2)$, where $q : \mathbb{R} \to V$. In this case, the classical space of states is $Y := T^*V = V \oplus V^*$. The equations of motion are Newton's equations $\ddot{q} = -m^2q$, which can be reduced to Hamilton's equations $\dot{q} = p$, $\dot{p} = -m^2q$. The algebra of classical observables is $C^\infty(Y)$, with Poisson bracket defined by $\{a, b\} = (a, b)$, $a, b \in Y^*$, where $(,)$ is the form on Y^* inverse to the natural symplectic form on Y. The hamiltonian H is determined (up to adding a constant) by the condition that the equations of motion are $\dot{f} = \{f, H\}$; in this case it is $H = (p^2 + m^2q^2)/2$.

The situation in the fermionic case is analogous, with some important differences which we will explain below. Namely, it is easy to compute that the equation of motion (i.e. the Euler-Lagrange equation) is $\dot{\psi} = M\psi$. The main difference with the bosonic case is that this equation is of first and not of second order, so the space of classical states is just ΠV (no momentum or velocity variables are introduced). Hence the algebra of classical observables is $C^\infty(\Pi V) = \Lambda V^*$. To define a Poisson bracket on this algebra, recall that ΠV has a natural "symplectic structure", defined by the *symmetric* form $(,)$ on V. Thus we can define a Poisson bracket on ΛV^* by the same formula as above: $\{a, b\} = (a, b)$. More precisely, $\{,\}$ is a unique skew symmetric (in the supersense) bilinear operation on ΛV^* which restricts to (a, b) for $a, b \in V^*$, and is a derivation with respect to each variable:

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)}b\{a, c\},$$

where $p(a)$ denotes the parity of a.

Now it is easy to see what should play the role of the Hamiltonian. More precisely, the definition with Legendre transform is not valid in our situation, since the Legendre transform was done with respect to the velocity variables, which we don't have in the fermionic case. On the other hand, as we discussed in §8, in the bosonic case the equation of motion $\dot{f} = \{f, H\}$ determines H uniquely, up to a constant. The situation is the same in the fermionic case. Namely, by looking at the equation of motion $\dot{\psi} = M\psi$, it is easy to see that the Hamiltonian equals $H = \frac{1}{2}(\psi, M\psi)$. In particular, if $M = 0$ (massless case), the Hamiltonian is zero (a characteristic property of topological field theories).

Now let us turn to quantum theory. In the bosonic case the algebra of quantum observables is a noncommutative deformation of the algebra $C^\infty(Y)$ in which the relation $\{a, b\} = (a, b)$ is replaced with its quantum analog $ab - ba = i(a, b)$ (up to the Planck constant factor which we will now ignore). In particular, the subalgebra of polynomial observables is the Weyl algebra $W(Y)$, generated by Y^* with this defining relation. By analogy with this, we must define the algebra of quantum observables in the fermionic case to be generated by V^* with the relation $ab + ba = i(a, b)$ (it deforms the relation $ab + ba = 0$ which defines ΛV^*). So we recall the following definition.

Definition 10.2. Let V be a vector space over a field k with a symmetric bilinear form Q. The Clifford algebra $Cl(V, Q)$ is generated by V with defining relations $ab + ba = Q(a, b)$, $a, b \in V$.

We see that the algebra of quantum observables should be $Cl(V_{\mathbb{C}}^*, i(,))$. Note that like in the classical case, this algebra is naturally $\mathbb{Z}/2$ graded, so that we have even and odd quantum observables.

Now let us see what should be the Hilbert space of quantum states. In the bosonic case it was $L^2(V)$, which is, by the well known Stone-von Neumann theorem, the unique irreducible unitary representation of $W(Y)$. By analogy with this, in the fermionic case the Hilbert space of states should be an irreducible even unitary representation of $Cl(V)$ on a supervector space $\mathcal{H}$.

The structure of the Clifford algebra $Cl(V^*)$ is well known. Namely, consider separately the cases when $\dim V$ is odd and even.

In the even case, $\dim V = 2d$, $Cl(V^*)$ is simple, and has a unique irreducible representation $\mathcal{H}$, of dimension 2^d. It is constructed as follows: choose a decomposition $V_{\mathbb{C}} = L \oplus L^*$, where L, L^* are Lagrangian subspaces. Then $\mathcal{H} = \Lambda L$, where $L \subset V_{\mathbb{C}}^*$ acts by multiplication and L^* by differentiation (multiplied by $-i$). The structure of the superspace on $\mathcal{H}$ is the standard one on the exterior algebra.

In the odd case, $\dim V = 2d+1$, choose a decomposition $V_{\mathbb{C}} = L \oplus L^* \oplus K$, where L, L^* are maximal isotropic, and K is a nondegenerate 1-dimensional subspace orthogonal to L and L^*. Let $\mathcal{H} = \Lambda(L \oplus K)$, where L, K act by multiplication and L^* by ($-i$ times) differentiation. This is a representation of $Cl(V^*)$ with a $\mathbb{Z}/2$ grading. This representation is not irreducible, and decomposes in a direct sum of two non-isomorphic irreducible representations $\mathcal{H}_+ \oplus \mathcal{H}_-$ (this is related to the fact that the Clifford algebra for odd $\dim V$ is not simple but is a sum of two simple algebras). However, this decomposition is not consistent with the $\mathbb{Z}/2$-grading, and therefore as superrepresentation, $\mathcal{H}$ is irreducible.

Now, it is easy to show that both in the odd and in the even case the space $\mathcal{H}$ carries a unique up to scaling Hermitian form, such that $V^* \subset V_{\mathbb{C}}^*$ acts by selfadjoint operators. This form is positive definite. So the situation is similar to the bosonic case for any $\dim V$.

Let us now see which operator on $\mathcal{H}$ should play the role of the Hamiltonian of the system. The most natural choice is to define the quantum Hamiltonian to be the obvious quantization of the classical Hamiltonian $H = \frac{1}{2}(\psi, M\psi)$. Namely, if ε_i is a basis of V^* and a_{ij} is the matrix of M in this basis, then one sets $\hat{H} = \frac{1}{2}\sum_{i,j} a_{ij}\varepsilon_i\varepsilon_j$. To compute this operator more explicitly, we will assume (without loss of generality) that the decomposition of $V_{\mathbb{C}}$ that we chose is stable under M. Let ξ_j be an eigenbasis of M in L (with eigenvalues im_j), and ∂_j be differentiations along the vectors of this basis. Then

$$\hat{H} = \sum_j m_j(\xi_j\partial_j - \partial_j\xi_j) = \sum_j m_j(2\xi_j\partial_j - 1).$$

This shows that if $\dim V$ is even then the partition function on the circle of length L for our theory is

$$Z = \mathrm{sTr}(e^{-L\hat{H}}) = \prod_j (e^{m_j L} - e^{-m_j L})$$

If the dimension of V is odd then the partition function is zero.

Now we would like to consider the fermionic analog of the Feynman-Kac formula. For simplicity consider the fully massive case, when $\dim V$ is even and $m_j \neq 0$ (i.e. M is nondegenerate). In this case, we have a unique up to scaling lowest eigenvector of $\hat{H}$, namely $\Omega = 1$.

Let $\psi(0) \in V \otimes \mathrm{End}(\mathcal{H})$ be the element corresponding to the action map $V^* \to \mathrm{End}(\mathcal{H})$, and $\psi(t) = e^{it\hat{H}}\psi(0)e^{-it\hat{H}}$. Also, let $< \psi(t_1)\cdots\psi(t_n) >$, $t_1 \geq \cdots \geq t_n$, be the correlation function for the free theory in the Lagrangian setting, taking values in $V^{\otimes n}$, so in this expression $\psi(t_i)$ is a formal symbol and not an operator.

Theorem 10.3. *(i) For the free theory on the line we have*

$$< \psi(t_1)\cdots\psi(t_n) >= (\Omega, \psi(t_1)\cdots\psi(t_n)\Omega).$$

(ii) For the free theory on the circle of length L we have

$$< \psi(t_1)\cdots\psi(t_n) >= \frac{\mathrm{sTr}(\psi(t_1)\cdots\psi(t_n)e^{-L\hat{H}})}{\mathrm{sTr}(e^{-L\hat{H}})}$$

Exercise. Prove this theorem. (The proof is analogous to Theorem 8.3 in the free case).

It should now be straightforward for the reader to formulate and prove the Feynman-Kac formula for an interacting theory which includes both bosonic and Fermionic massive fields. We leave this as an instructive exercise.