Final Examination

18.303 Linear Partial Differential Equations

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Total points: 100

1 Rules [requires student signature!]

1. I will use only pencils, pens, erasers, and straight edges to complete this exam.
2. I will NOT use calculators, notes, books or other aides.

Signature: ____ Date: ____________.

Please hand in this question sheet with your solutions following the exam.

2 Note

Work on problems (and sub-parts) in any order; just be sure to label the question. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given on the next two pages, without proof, on any question in the exam.

3 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition for sines and cosines: for any $L > 0$ (e.g. $L = 1$, π, $\pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 rJ_n\left(j_{n,m}r\right)J_k\left(j_{k,l}r\right)dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r \left(J_n \left(j_{n,m} r\right)\right)^2 dr \qquad (> 0)$$

A useful result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

The Jacobian determinant of the change of variable $(r, s) \to (x, t)$ is

$$\frac{\partial (x, t)}{\partial (r, s)} = \det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

Rayleigh Quotient:

$$R(v) = \frac{\int\int\int_D \nabla v \cdot \nabla v dV}{\int\int\int_D v^2 dV} = \frac{\int\int\int_D |\nabla v|^2 dV}{\int\int\int_D v^2 dV}$$

Trig identities:

$$\begin{aligned} \sin a \sin b &= \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right) \\ \cos a \cos b &= \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right) \\ \sin(a+b) &= \sin a \cos b + \sin b \cos a \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \end{aligned}$$

The spatial Fourier Transform of $u(x,t)$ and $f(x)$ are defined as

$$\begin{aligned} \overline{U}(\omega, t) &= \mathcal{F}\left[u(x,t)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx \\ F(\omega) &= \mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \end{aligned}$$

The Inverse Fourier Transforms of $\overline{U}(\omega, t)$ and $F(\omega)$ are defined as

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\overline{U}(\omega,t)\right](x) = \int_{-\infty}^{\infty} \overline{U}(\omega,t) e^{-i\omega x} d\omega \\ f(x) &= \mathcal{F}^{-1}\left[F(\omega)\right](x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \end{aligned}$$

The IFT of a Gaussian is

$$\mathcal{F}^{-1}\left[e^{-\alpha\omega^2}\right] = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha} \tag{2}$$

where α can involve constants or variables, but must be independent of ω and x.

4 Questions

4.1 Question 1

[30 marks, suggested time: 30-40 mins]

(a) [10 marks] Solve Laplace's Equation on the quarter unit disc,

$$\nabla^2 u(r,\theta) = 0$$

with BCs

$$u(1,\theta) = g(\theta), \qquad u(0,\theta) \text{ bounded}, \qquad 0 < \theta < \pi/2,$$
$$u(r,0) = 0, \qquad u\left(r, \frac{\pi}{2}\right) = 0, \qquad 0 < r < 1.$$

Be sure to use any relevant given information to save time.

(b) [12 marks] Solve the Heat Problem on the unit quarter disc

$$v_t = \nabla^2 v, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

subject to inhomogeneous BCs

$$v(1,\theta,t) = g(\theta), \qquad v(0,\theta,t) \text{ bounded}, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$
$$v(r,0,t) = 0, \qquad v\left(r, \frac{\pi}{2}, t\right) = 0, \qquad 0 < r < 1, \qquad t > 0,$$

and initial condition

$$v(r,\theta,0) = f(r,\theta), \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

Your solution will have coefficients in terms of integrals involving $f(r,\theta)$.

(c) [8 marks] Prove the solution to (b) is unique. Hint: The steps follow those for the 1D rod, but you'll need to use a result derived from the Divergence Theorem (on the given page) instead of integration by parts. You don't need to consider r, θ: denoting the region by D and using dV will work fine.

Solution: (a) Separate variables as

$$u(r,\theta) = R(r)\,H(\theta)$$

so that the PDE becomes

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{H}\frac{d^2 H}{d\theta^2} = \lambda$$

where λ is constant, since the l.h.s. depends only on r and the r.h.s. only on θ. Separating the BCs gives

$$\begin{aligned} u(r,0) &= 0 \Rightarrow H(0) = 0 \\ u\left(r,\frac{\pi}{2}\right) &= 0 \Rightarrow H(\pi/2) = 0 \end{aligned}$$

We solve for $H(\theta)$ first:

$$H'' + \lambda H = 0; \qquad H(0) = 0 = H(\pi/2).$$

Using the given information, we have

$$H(\theta) = \sin\left(\frac{n\pi\theta}{\pi/2}\right) = \sin(2n\theta), \qquad n = 1, 2, 3...$$

and $\lambda = 4n^2$. Thus, the equation for $R(r)$ is

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (2n)^2 R = 0$$

Try $R = r^\alpha$, so that $\alpha = \pm 2n$:

$$R = c_1 r^{2n} + c_2 r^{-2n}$$

For $R(0)$ to be bounded, we must have $c_2 = 0$. Putting things together gives

$$u_n(r,\theta) = R(r)\,H(\theta) = r^{2n}\sin(2n\theta), \qquad n = 1, 2, 3...$$

Using superposition, the general solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n u_n(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin(2n\theta)$$

where the A_n's are found using the BC at $r = 1$ and orthogonality,

$$g(\theta) = u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating from $\theta = 0$ to $\theta = \pi/2$ gives

$$\int_0^{\pi/2} g(\theta)\sin(2m\theta)\,d\theta = A_m \frac{\pi}{2}\frac{1}{2}$$

Thus

$$A_m = \frac{4}{\pi}\int_0^{\pi/2} g(\theta)\sin(2m\theta)\,d\theta$$

(b) First, let

$$V(r,\theta,t) = v(r,\theta,t) - u(r,\theta)$$

where $u(r,\theta)$ was found in part (a). Then $V(r,\theta,t)$ satisfies the homogeneous problem,

$$V_t = \nabla^2 V, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

$$\begin{aligned} V(1,\theta,t) &= 0, \qquad V(0,\theta,t) \text{ bounded}, \qquad 0 < r < 1, \qquad t > 0, \\ V(r,0,t) &= 0, \qquad V\left(r,\frac{\pi}{2},t\right) = 0, \qquad 0 < \theta < \pi/2, \qquad t > 0, \end{aligned}$$

and initial condition

$$V(r,\theta,0) = f(r,\theta) - u(r,\theta), \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

We separate variables as

$$V(r,\theta,t) = R(r)\,H(\theta)\,T(t)$$

so that the PDE becomes

$$\frac{T'}{T} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2 H}\frac{d^2H}{d\theta^2} = -\lambda$$

where λ is constant since the lhs depends only on t, and the rhs on r, θ. Since the solution must decay, we expect $\lambda > 0$. Or we could argue this from general theory. From the middle equation, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{H}\frac{d^2H}{d\theta^2} = \mu$$

again, since the lhs depends on r only and the rhs on θ only. Separating the BCs yields

$$V(1,\theta,t) = 0 \Rightarrow R(1) = 0$$

$$V(1,\theta,t) \text{ bounded } \Rightarrow |R(0)| < \infty$$

$$\begin{aligned} V(r,0,t) &= 0 \Rightarrow H(0) = 0, \\ V\left(r,\frac{\pi}{2},t\right) &= 0 \Rightarrow H\left(\frac{\pi}{2}\right) = 0. \end{aligned}$$

The problem for $H(\theta)$ is the same as before, thus

$$H(\theta) = \sin(2n\theta)$$

and $\mu = 4n^2$. Thus

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - (2n)^2\right) R = 0$$

and hence the general solution is

$$R(r) = c_1 J_{2n}\left(\sqrt{\lambda} r\right) + c_2 Y_{2n}\left(\sqrt{\lambda} r\right)$$

Since $R(0)$ must be bounded, $c_2 = 0$. The other BC is

$$1 = R(1) = c_1 J_{2n}\left(\sqrt{\lambda}\right)$$

Thus $\lambda_{nm} = j_{2n,m}^2$, where $j_{2n,m}$ is the m'th zero of J_{2n}.

Thus, for $n, m = 1, 2, 3, \ldots$

$$V_{nm}(r, \theta, t) = J_{2n}(r j_{2n,m}) \sin(2n\theta) e^{-\lambda_{nm} t}$$

solves the PDE and BCs. To solve the IC, we sum over n, m and use superposition,

$$V(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} J_{2n}(r j_{2n,m}) \sin(2n\theta) e^{-\lambda_{nm} t}$$

where the A_{nm}'s are found from the IC and orthogonality:

$$f(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} J_{2n}(r j_{2n,m}) \sin(2n\theta)$$

Multiplying by $r J_{2k}(r j_{2k,l}) \sin(2k\theta)$ and integrating in r, θ, we have

$$\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (f(r, \theta) - u(r, \theta)) r J_{2k}(r j_{2k,l}) \sin(2k\theta)\, d\theta = A_{kl} \frac{\pi}{4} \int_{r=0}^{1} r \left(J_{2k}(r j_{2k,l})\right)^2 dr$$

Thus

$$A_{kl} = \frac{4}{\pi} \frac{\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (f(r, \theta) - u(r, \theta)) r J_{2k}(r j_{2k,l}) \sin(2k\theta)\, d\theta}{\int_{r=0}^{1} r \left(J_{2k}(r j_{2k,l})\right)^2 dr}$$

Finally,

$$v(r, \theta, t) = V(r, \theta, t) + u(r, \theta)$$

(c) Take 2 solutions v_1, v_2. Define the difference $h = v_1 - v_2$. Note that h satisfies

$$h_t = \nabla^2 h, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

$$\begin{aligned} h(1, \theta, t) &= 0, \qquad h(0, \theta, t) \text{ bounded}, \qquad 0 < r < 1, \qquad t > 0, \\ h(r, 0, t) &= 0, \qquad h\left(r, \frac{\pi}{2}, t\right) = 0, \qquad 0 < \theta < \pi/2, \qquad t > 0, \end{aligned}$$

$$h(r, \theta, 0) = 0, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

Define the mean square difference between solutions,

$$\bar{V}(t) = \int\int_D h^2 dV \geq 0$$

Differentiate in time,

$$\begin{aligned}\frac{d\bar{V}(t)}{dt} &= \int\int_D 2hh_t dV = \int\int_D 2h\nabla^2 h dV \\ &= -2\int\int_D |\nabla v|^2 \, dV + 2\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS\end{aligned}$$

But $v = 0$ on the boundary, so that

$$\frac{d\bar{V}(t)}{dt} = -2\int\int_D |\nabla v|^2 \, dV \leq 0$$

Note that at $t = 0$,

$$\bar{V}(0) = \int\int_D (f(r,\theta) - f(r,\theta))^2 \, dV = 0$$

Thus, $\bar{V}(t)$ is a non-negative, non-increasing function, that starts at zero. Hence $\bar{V}(t) = 0$ for all time, which implies by continuity that $h(r,\theta,t) = 0$ for all r, θ, t. Hence $v_1 = v_2$, and the solution to (b) is unique.

4.2 Question 2

[15 marks, suggested time 20 mins]

Suppose you shake a rope of length 1 sinusoidally with specified and fixed frequency ω on one end ($x = 1$), while the other end ($x = 0$) is attached to a frictionless coupling that can oscillate vertically. We model the problem using the 1D wave equation

$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0, \tag{3}$$

subject to the BCs

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0, \tag{4}$$

$$u(1,t) = \cos\omega t, \qquad t > 0. \tag{5}$$

We assume the initial condition is that you hold the rope at $x = 1$ away from its rest position, but give it zero initial velocity,

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < 1. \tag{6}$$

NOTE: the value of ω is fixed and a parameter of the problem - it is not something you solve for!

(a) [5 marks] Find a particular solution to the PDE (3) and BCs (4) and (5). For what values of ω will this not work? These are resonant frequencies - you may assume ω is not one of these. Hint: try $u_{SS}(x,t) = X(x)\cos\omega t$.

(b) [10 marks] Use your solution in (a) to help you find the full solution to the PDE (3), BCs (4) and (5), and ICs (6). Find $u(0,t)$, the motion of the coupling. Hint: obtain a wave problem with homogeneous BCs and use D'Alembert (you don't have to derive D'Alembert). Extend functions appropriately to satisfy the BCs, and explain why this works: at $x = 0$ this is straightforward; at $x = 1$ this takes a little thought (so move on if you don't get it). Also, you don't have to substitute for the functions in D'Alembert, just say how you'll extend them. Substituting them before you extend them won't work.

Solution: (a) Sub $u_{SS}(x,t) = X(x)\cos\omega t$ into the PDE,

$$-\omega^2 X(x) = X''$$

and hence

$$X = A\cos\omega x + B\sin\omega x$$

Insert into the BCs:

$$X'(0) = 0, \qquad X(1) = 1$$

and hence

$$\begin{aligned} X'(0) &= B = 0 \\ X(1) &= A\cos\omega = 1 \end{aligned}$$

Thus

$$A = 1/\cos\omega$$

Thus

$$u_{SS}(x,t) = \frac{\cos\omega x}{\cos\omega}\cos\omega t$$

(b) Define

$$V(x,t) = u(x,t) - u_{SS}(x,t)$$

The problem for $V(x,t)$ is

$$\begin{aligned} V_{tt} &= V_{xx} \\ \frac{\partial V}{\partial x}(0,t) &= 0 \\ V(1,t) &= 0 \end{aligned}$$

and the ICs are

$$V(x,0) = u(x,0) - u_{SS}(x,0) = x - \frac{\cos\omega x}{\cos\omega} = f(x)$$

$$\frac{\partial V}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial u_{SS}}{\partial t}(x,0) = 0 - 0 = 0$$

thus D'Alembert's formula is

$$V(x,t) = \frac{1}{2}\left(\tilde{f}(x-t) + \tilde{f}(x+t)\right)$$

To satisfy the BC at $x = 0$, extend $f(s)$ to be even. To satisfy the BC at $x = 1$, extend $f(s)$ to be anti-symmetric about $s = 1$, i.e.

$$\tilde{f}(s) = \begin{cases} f(s), & 0 < s < 1, \\ f(-s), & -1 < s < 0, \\ -f(2-s), & 1 < s < 2, \\ -f(2+s), & -2 < s < -1. \end{cases}$$

Then extend this function to be $4-$periodic. The solution at $x = 0$ is

$$\begin{aligned} u(0,t) &= \frac{\cos\omega t}{\cos\omega} + \frac{1}{2}\left(\tilde{f}(-t) + \tilde{f}(t)\right) \\ &= \frac{\cos\omega t}{\cos\omega} + \tilde{f}(t) \end{aligned}$$

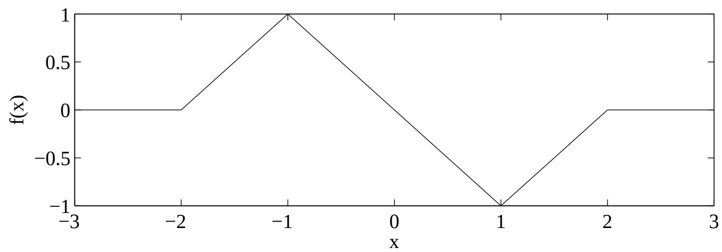

Figure 1: Sketch of $f(x)$.

4.3 Question 3

[20 marks, suggested time 20 mins]

Consider the quasi-linear PDE

$$\frac{\partial u}{\partial t} + (1 - |u|)\frac{\partial u}{\partial x} = 0; \qquad u(x,0) = f(x)$$

where

$$f(x) = \begin{cases} x+2, & -2 \le x \le -1, \\ -x, & -1 \le x \le 1, \\ x-2, & 1 \le x \le 2, \\ 0, & x > 2 \text{ or } x < -2 \end{cases}$$

The function $f(x)$ is plotted in Figure 1.

(a) [8 points] By writing the PDE in the form $(A, B, C) \cdot (u_t, u_x, -1) = 0$, find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\partial t/\partial r$, $\partial x/\partial r$, $\partial u/\partial r$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition, write down the IC for u at $r = 0$, in terms of s.

(b) [6 points] At what time t_{sh} and location(s) x_{sh} does your parametric solution break down? Hint: you may assume

$$\frac{d}{ds}|f(s)| = \frac{df}{ds}\frac{f(s)}{|f(s)|},$$

and there may be more than one breakdown location.

(c) [6 points] Plot $u(x,t)$ vs. x when $t = 1/2$. Hint: $f(x)$ is piecewise linear, so you may find it useful to construct a table for $s = -2, -1, 0, 1, 2$ and u and x at $t = 1/2$. Note: $t = 1/2$ is not necessarily the breakdown time.

Solution: (a) We write the PDE as

$$(1, 1 - |u|, 0) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1\right) = 0$$

so that the parametric equations are

$$\frac{\partial t}{\partial r} = 1, \qquad \frac{\partial x}{\partial r} = 1 - |u|, \qquad \frac{\partial u}{\partial r} = 0$$

Thus with the ICs,

$$t = r, \qquad u = f(s), \qquad x = (1 - |f(s)|)\, r + s$$

(b) The Jacobian is

$$\det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = -x_s = -\left(-f'(s) \frac{f(s)}{|f(s)|} \right) r - 1 = 0$$

if

$$t = r = \frac{|f(s)|}{f(s) f'(s)}$$

Noting that

$$f'(x) = \begin{cases} 1, & -2 \le x \le -1, \\ -1, & -1 \le x \le 1, \\ 1, & 1 \le x \le 2, \\ 0, & x > 2 \text{ or } x < -2 \end{cases}$$

we obtain the first breakdown time,

$$t = \frac{1}{2 f(s) f'(s)} = \frac{1}{\max(f(s))}, \frac{-1}{\min(f(s))} = 1$$

Breakdown occurs along $s = \pm 1$,

$$\begin{aligned} x &= (1 - |f(1)|)\, 1 + s = s = 1 \\ x &= (1 - |f(-1)|)\, 1 + s = s = -1 \end{aligned}$$

(c) Make a table

$t = \frac{1}{2}$:

s	-2	-1	0	1	2
$u = f(s)$	0	1	0	-1	0
x	$1/2 - 2 = -3/2$	-1	$1/2$	1	$1/2 + 2 = 5/2$

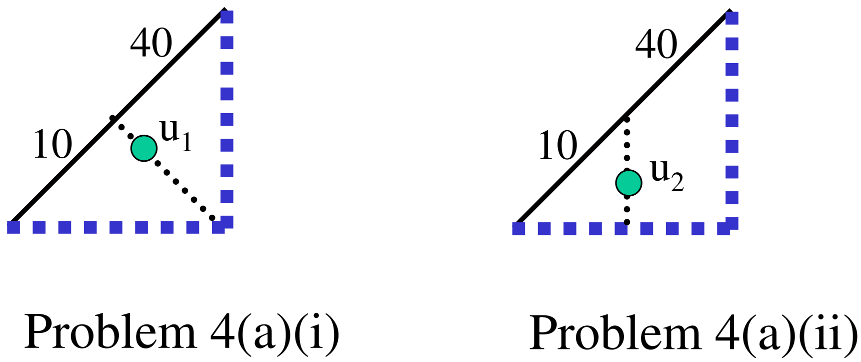

Figure 2: Diagrams for part (a)(i) and (a)(ii).

4.4 Question 4

[15 marks, suggested time 20 mins]

(a) [7 marks] Consider the boundary value problem on the isosceles right angled triangle of side length 1,

$$\nabla^2 v = 0, \qquad 0 < y < x, \qquad 0 < x < 1$$

subject to the BCs

$$\begin{aligned} \frac{\partial v}{\partial x}(1, y) &= 0, \qquad 0 < y < 1 \\ \frac{\partial v}{\partial y}(x, 0) &= 0, \qquad 0 < x < 1 \\ v(x, x) &= 10, \qquad 0 < x < 1/2 \\ v(x, x) &= 40, \qquad 1/2 < x < 1 \end{aligned}$$

(i) Give a symmetry argument to find $v(x, 1 - x)$ for $1/2 < x < 1$. See diagram in Figure 2. [1/2 point for answer, 3.5 for argument]

(ii) Give a symmetry argument to find $v(1/2, y)$ for $0 < y < 1/2$. See diagram in Figure 2. [1/2 point for answer, 2.5 for argument]

Solution: (i) Rotate about $y = 1 - x$, add, get $u = 50$ on boundary, thus $u = 50$ everywhere. Hence $2u_1 = 50$ and $u_1 = 25$.

(ii) Now that you know $u = 25$ on $y = 1 - x$, slice that triangle off, rotate about $x = 1/2$, add, get $u = 35$ on bdry and hence everywhere, thus, $u_1 = 35/2$.

Question 4 (continued)

(b) [8 marks] Find an eigenvalue λ and corresponding eigenfunction v for the right triangle

$$D = \left\{(x, y) : 0 < y < \sqrt{2}x, \quad 0 < x < 1\right\}$$

with side lengths 1 and $\sqrt{2}$. v and λ satisfy the Sturm-Liouville Problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D. \end{aligned}$$

Hint: you may use the eigenfunctions derived in-class for the rectangle, without derivation. You may find constructing a table useful for $2m^2 + n^2$ $(n, m = 1, 2, 3)$

(c) [5 BONUS marks] Find a function that is zero on the boundary of the triangle, nonzero and smooth on the interior, and use it to obtain an upper bound on the smallest eigenvalue of the triangle in (b). You don't have to evaluate the integrals; just set them up.

Solution: Eigenfunctions on rectangle are

$$v_{mn} = \sin(m\pi x)\sin\left(\frac{n\pi y}{\sqrt{2}}\right), \qquad \lambda_{mn} = \frac{\pi^2}{2}\left(2m^2 + n^2\right)$$

Make table for $2m^2 + n^2$,

$m \backslash n$	1	2
1	3	5
2	5	
3		

Thus 5 repeats, and both have eigenvalue $\lambda_{21} = \lambda_{12} = 5\pi^2/2$. We add

$$v = v_{21} + Av_{12}$$

We know these are both zero on the vertical and horizontal side. We find A such that they are zero on $y = \sqrt{2}x$,

$$\begin{aligned} 0 &= v\left(x, \sqrt{2}x\right) \\ &= v_{21}\left(x, \sqrt{2}x\right) + Av_{12}\left(x, \sqrt{2}x\right) \\ &= \sin(2\pi x)\sin(\pi x) + A\sin(\pi x)\sin(2\pi x) \end{aligned}$$

Thus $A = -1$.

(c) Use $v(x, y) = y(x-1)\left(\sqrt{2}x - y\right)$. Use Rayleigh Quotient, will be upper bound on λ_1.

4.5 Question 5

[20 marks, suggested time 25 mins]

Consider the Heat Equation on an infinite strip,

$$u_t = u_{xx} + u_{yy}, \quad -\infty < x < \infty, \quad 0 < y < 1, \quad t > 0, \tag{7}$$

subject to homogeneous Type II (insulated) BCs along $y = 0, 1$:

$$\frac{\partial u}{\partial y}(x, 0, t) = 0 = \frac{\partial u}{\partial y}(x, 1, t), \quad -\infty < x < \infty, \quad t > 0. \tag{8}$$

Assume the initial temperature distribution is separable

$$u(x, y, 0) = f(x) g(y). \tag{9}$$

(a) [4 marks] Separate as $u(x, y, t) = v(x, t) Y(y)$ and obtain a Sturm-Liouville Problem for $Y(y)$. Obtain the eigenfunctions, and eigenvalues. State the problem for $v(x, t)$.

(b) [3 marks] For each eigenfunction $Y_n(y)$, make the transformation $V(x, t) = e^{\beta t} v(x, t)$ to obtain

$$\begin{aligned} V_t &= V_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ V(x, 0) &= f(x). \end{aligned}$$

You'll need to find β to obtain this - it will depend on n.

(c) [5 marks] Solve for $V(x, t)$ using the Fourier Transform (defined on page 3). Solve for the transform $\bar{V}(\omega, t)$. We did this in class, but please show your steps. Invert to find $V(x, t)$.

(d) [5 marks] Put the solution back together to obtain $u_n(x, y, t) = v_n(x, t) Y_n(y)$, which satisfies the PDE (7) and BCs (8). Use these to obtain the full solution $u(x, y, t)$ that satisfies the IC (9). Compute any coefficients in terms of integrals of $g(y)$.

(e) [3 marks] Find $\partial u / \partial x (x, y, t)$ and set $x = 0$. What property must $f(x)$ have so that

$$\frac{\partial u}{\partial x}(0, y, t) = 0?$$

Solution: (a) Separate as $u(x, y, t) = v(x, t) Y(y)$:

$$\frac{v_t - v_{xx}}{v} = \frac{Y''}{Y} = -\lambda$$

where λ is constant since lhs dep only on t, x and rhs on y. Separate BCs:

$$\begin{aligned} \frac{\partial u}{\partial y}(x, 0, t) &= 0 \Rightarrow Y'(0) = 0 \\ \frac{\partial u}{\partial y}(x, 1, t) &= 0 \Rightarrow Y'(1) = 0 \end{aligned}$$

Thus

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(1)$$

The eigenvalues and eigen-functions are (given)

$$Y_n(y) = \cos n\pi y, \qquad \lambda_n = (n\pi)^2, \qquad n = 0, 1, 2, 3...$$

Thus,

$$v_t = v_{xx} + \lambda_n v$$

(b) Make the transformation $V(x,t) = e^{-\lambda_n t} v(x,t)$ to obtain

$$\begin{aligned} V_t &= e^{-\lambda_n t} v_t - \lambda_n e^{-\lambda_n t} v \\ V_{xx} &= e^{-\lambda_n t} v_{xx} = e^{-\lambda_n t} v_t - \lambda_n e^{-\lambda_n t} v = V_t \\ V(x,0) &= v(x,0) = f(x) \end{aligned}$$

Thus

$$\begin{aligned} V_t &= V_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ V(x,0) &= f(x). \end{aligned}$$

(c) Apply the FT,

$$\frac{\partial \bar{V}}{\partial t} = -\omega^2 \bar{V}$$

and integrate to obtain

$$\bar{V}(\omega, t) = c(\omega) e^{-\omega^2 t}$$

Apply FT to IC

$$F(\omega) = \bar{V}(\omega, 0) = c(\omega)$$

Thus

$$\bar{V}(\omega, t) = F(\omega) e^{-\omega^2 t}$$

Note that the IFT of $e^{-\omega^2 t}$ is $\sqrt{\frac{\pi}{t}} e^{-x^2/4t}$. Apply Convolution Theorem:

$$\begin{aligned} V(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{t}} \exp\left(-\frac{(x-s)^2}{4t}\right) ds \\ &= \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi t}} \exp\left(-\frac{(x-s)^2}{4t}\right) ds \end{aligned}$$

(d) Thus,

$$\begin{aligned} u_n(x,y,t) &= e^{-\lambda_n t} V(x,t) Y_n(y) \\ &= V(x,t) \cos(n\pi y) e^{-\lambda_n t} \end{aligned}$$

The general solution is obtained from superposition:

$$u(x,y,t) = V(x,t)\left(A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi y)\, e^{-\lambda_n t}\right)$$

We find the A_n's using orthogonality:

$$f(x)\, g(y) = V(x,0)\left(A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi y)\right)$$

and hence

$$g(y) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi y)$$

so that

$$A_0 = \int_0^1 g(y)\, dy, \qquad A_n = \frac{1}{2}\int_0^1 g(y)\cos(n\pi y)\, dy$$

(e) We have

$$\frac{\partial u}{\partial x}(0,y,t) = V_x(0,t)\left(A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi y)\, e^{-\lambda_n t}\right)$$

Thus

$$0 = V_x(0,t) = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi t}}\left(\frac{2s}{4t}\right)\exp\left(-\frac{s^2}{4t}\right) ds$$

and hence $f(s)$ must be even.