Lecture 12
The Laplace Method

We begin with integrals of the form

$$I(\lambda) = \int_a^b e^{-\lambda v(x)} h(x) dx, \qquad (8.8)$$

where $v(x)$ and $h(x)$ are independent of the parameter λ. The variable of integration x is real. We shall show how to find the asymptotic form of $I(\lambda)$ as $\lambda \to \infty$.

We note that the integrand in the integral above is maximum when v is minimum. Indeed, let x_0 be the point inside the interval $[a, b]$ at which $v(x)$ is minimum. Then the integrand at x_0 is exponentially larger than that at any other points in the region of integration. As a result, to evaluate the integral of (8.8) approximately, it is crucial to approximate $v(x)$ and $h(x)$ accurately near x. This also means that we need not be concerned with the integrand away from x_0. This is because the dominant contribution to the integral comes from a very small region near x_0.

Problem for the Reader:

Find the leading asymptotic term of the integral

$$I(\lambda) = \int_0^\infty \exp(-\lambda \sinh^2 x) dx, \lambda \gg 1. \qquad (8.9)$$

Answer

Comparing (8.9) with (8.8), we make the identification

$$v(x) = \sinh^2 x,$$

and

$$h(x) = 1.$$

The function $\sinh^2 x$ is always positive, and is minimum at $x = 0$. Thus it suffices to represent $\sinh^2 x$ accurately near the origin, where

$$\sinh^2 x \approx x^2.$$

We therefore have

$$I(\lambda) \approx \int_{-\infty}^{\infty} e^{-\lambda x^2} dx.$$

Let

$$\rho \equiv \sqrt{\lambda}\, x,$$

and

$$dx = d\rho/\sqrt{\lambda}.$$

With this change of variable the exponent is independent of λ. Then we have

$$I(\lambda) \approx \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-\rho^2} d\rho = \sqrt{\frac{\pi}{\lambda}}. \qquad (8.10)$$

In the above, we have made use of the fact that the Gaussian integral $\int_{-\infty}^{\infty} e^{-\rho^2} d\rho$ is equal to $\sqrt{\pi}$, as is given in Appendix A of this chapter.

If one makes a small mistake in an approximation, the answer one gets can be far off the mark. Thus it is often useful to have a quick answer. We shall show how to get a rough estimate of the asymptotic form of the integral $I(\lambda)$ without going through a great deal of calculations performed.

The integrand of $I(\lambda)$ drops by a factor of e when

$$\lambda x^2 = 1,$$

or

$$x = 1/\sqrt{\lambda}.$$

Thus $1/\sqrt{\lambda}$ is roughly the width of the region which gives the dominant contributions to the integral. The width $1/\sqrt{\lambda}$ of the region of dominant contribution is the same factor relating dx with dt. We shall say that the scale of x is $1/\sqrt{\lambda}$.

The integral $I(\lambda)$ is roughly equal to its integrand at $x = 0$ times the width of the region of dominant contributions. Since the integrand at $x = 0$ is equal to unity, $I(\lambda)$ is of the order of $1/\sqrt{\lambda}$. This produces the answer (8.10) up to a multiplicative constant. We express this estimate as

$$I(\lambda) = O\left(1/\sqrt{\lambda}\right).$$

For certain problems such an estimate is already adequate for the purpose. If so, we are spared the chore of a tedious calculation. The factor missing in the estimate is the multiplicative constant $\sqrt{\pi}$, which is the value of the Gaussian integral.

Homework problems in Chapter 8 due Nov 1,04:
Problem 2 (The integrals are given on p. 218);
Problem 3;
Find also the entire asymptotic series of each of the integrals in Problem 3.