LECTURE 5
Finite fields

5.1. The finite field method

In this lecture we will describe a method based on finite fields for computing the characteristic polynomial of an arrangement defined over $\mathbb{Q}$. We will then discuss several interesting examples. The main result (Theorem 5.15) is implicit in the work of Crapo and Rota [**9**, §17]. It was first developed into a systematic tool for computing characteristic polynomials by Athanasiadis [**1**][**2**], after a closely related but not as general technique was presented by Blass and Sagan [**6**].

Suppose that the arrangement $\mathcal{A}$ is defined over $\mathbb{Q}$. By multiplying each hyperplane equation by a suitable integer, we may assume $\mathcal{A}$ is defined over $\mathbb{Z}$. In that case we can take coefficients modulo a prime p and get an arrangement $\mathcal{A}_q$ defined over the finite field $\mathbb{F}_q$, where $q = p^r$. We say that $\mathcal{A}$ has *good reduction* mod p (or over $\mathbb{F}_q$) if $L(\mathcal{A}) \cong L(\mathcal{A}_q)$.

For instance, let $\mathcal{A}$ be the affine arrangement in $\mathbb{Q}^1 = \mathbb{Q}$ consisting of the points 0 and 10. Then $L(\mathcal{A})$ contains three elements, viz., $\mathbb{Q}$, $\{0\}$, and $\{10\}$. If $p \neq 2, 5$ then 0 and 10 remain distinct, so $\mathcal{A}$ has good reduction. On the other hand, if $p = 2$ or $p = 5$ then $0 = 10$ in $\mathbb{F}_p$, so $L(\mathcal{A}_p)$ contains just two elements. Hence $\mathcal{A}$ has bad reduction when $p = 2, 5$.

Proposition 5.13. *Let $\mathcal{A}$ be an arrangement defined over $\mathbb{Z}$. Then $\mathcal{A}$ has good reduction for all but finitely many primes p.*

Proof. Let $H_1, \dots, H_j$ be affine hyperplanes, where H_i is given by the equation $v_i \cdot x = a_i$ ($v_i, a_i \in \mathbb{Z}^n$). By linear algebra, we have $H_1 \cap \cdots \cap H_j \neq \emptyset$ if and only if

$$\text{rank} \begin{bmatrix} v_1 & a_1 \\ \vdots & \vdots \\ v_j & a_j \end{bmatrix} = \text{rank} \begin{bmatrix} v_1 \\ \vdots \\ v_j \end{bmatrix}. \tag{36}$$

Moreover, if (36) holds then

$$\dim(H_1 \cap \cdots \cap H_j) = n - \text{rank} \begin{bmatrix} v_1 \\ \vdots \\ v_j \end{bmatrix}.$$

Now for any $r \times s$ matrix A, we have $\mathrm{rank}(A) \geq t$ if and only if some $t \times t$ submatrix B satisfies $\det(B) \neq 0$. It follows that $L(\mathcal{A}) \not\cong L(\mathcal{A}_p)$ if and only if at least one member S of a certain finite collection $\mathcal{S}$ of subsets of integer matrices B satisfies the following condition:

$$(\forall B \in S)\ \det(B) \neq 0 \text{ but } \det(B) \equiv 0 \,(\mathrm{mod}\, p).$$

This can only happen for finitely many p, viz., for certain B we must have $p | \det(B)$, so $L(\mathcal{A}) \cong L(\mathcal{A}_p)$ for p sufficiently large. $\square$

The main result of this section is the following. Like many fundamental results in combinatorics, the proof is easy but the applicability very broad.

Theorem 5.15. *Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^n$, and suppose that $L(\mathcal{A}) \cong L(\mathcal{A}_q)$ for some prime power q. Then*

$$\begin{aligned} \chi_{\mathcal{A}}(q) &= \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right) \\ &= q^n - \# \bigcup_{H \in \mathcal{A}_q} H. \end{aligned}$$

Proof. Let $x \in L(\mathcal{A}_q)$ so $\#x = q^{\dim(x)}$. Here $\dim(x)$ can be computed either over $\mathbb{Q}$ or F_q. Define two functions $f, g : L(\mathcal{A}_q) \to \mathbb{Z}$ by

$$\begin{aligned} f(x) &= \#x \\ g(x) &= \#\left(x - \bigcup_{y > x} y\right). \end{aligned}$$

In particular,

$$g(\hat{0}) = g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right).$$

Clearly

$$f(x) = \sum_{y \geq x} g(y).$$

Let μ denote the Möbius function of $L(\mathcal{A}) \cong L(\mathcal{A}_q)$. By Möbius inversion (Theorem 1.1),

$$\begin{aligned} g(x) &= \sum_{y \geq x} \mu(x,y) f(y) \\ &= \sum_{y \geq x} \mu(x,y) q^{\dim(y)}. \end{aligned}$$

Put $x = \hat{0}$ to get

$$g(\hat{0}) = \sum_{y} \mu(y) q^{\dim(y)} = \chi_{\mathcal{A}}(q).$$

$\square$

For the remainder of this lecture, we will be concerned with applications of Theorem 5.15 and further interesting examples of arrangements.

Example 5.12. Let G be a graph with vertices $1, 2, \ldots, n$, so

$$Q_{\mathcal{A}_G}(x) = \prod_{ij \in E(G)} (x_i - x_j).$$

Then by Theorem 5.15,

$$\begin{aligned} \chi_{\mathcal{A}_G}(q) &= q^n - \#\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_1^n \,:\, \alpha_i = \alpha_j \text{ for some } ij \in E(G)\} \\ &= \#\{(\beta_1, \ldots, \beta_n) \in \mathbb{F}_q^n \,:\, \beta_i \neq \beta_j \;\forall\, ij \in E(G)\} \\ &= \chi_G(q), \end{aligned}$$

in agreement with Theorem 2.7. Note that this equality holds for *all* prime powers q, not just for p^m with $p \gg 0$. This is because the matrix with rows $e_i - e_j$, where $ij \in E(G)$ and e_i is the ith unit coordinate vector in $\mathbb{Q}^n$, is *totally unimodular*, i.e., every minor (determinant of a square submatrix) is $0, \pm 1$. Hence the nonvanishing of a minor is independent of the ambient field.

A very interesting class of arrangements, including the braid arrangement, is associated with root systems, or more generally, finite reflection groups. We will simply mention some basic results here without proof. A *root system* is a finite set R of nonzero vectors in $\mathbb{R}^n$ satisfying certain properties that we will not give here. (References include [**4**][**7**][**12**].) The *Coxeter arrangement* $\mathcal{A}(R)$ consists of the hyperplanes $\alpha \cdot x = 0$, where $\alpha \in R$. There are four infinite (irreducible) classes of root systems (all in $\mathbb{R}^n$):

$$\begin{aligned} A_{n-1} &= \{e_i - e_j \,:\, 1 \leq i < j \leq n\} = \mathcal{B}_n \\ D_n &= \{e_i - e_j, e_i + e_j \,:\, 1 \leq i < j \leq n\} \\ B_n &= D_n \cup \{e_i \,:\, 1 \leq i \leq n\} \\ C_n &= D_n \cup \{2e_i \,:\, 1 \leq i \leq n\}. \end{aligned}$$

We should really regard A_{n-1} as being a subset of the space

$$\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \,:\, \sum \alpha_i = 0\} \cong \mathbb{R}^{n-1}.$$

We thus obtain the following Coxeter arrangements. In all cases $1 \leq i < j \leq n$ and $1 \leq k \leq n$.

$$\begin{aligned} \mathcal{A}(A_{n-1}) = \mathcal{B}_n &\;:\; x_i - x_j = 0 \\ \mathcal{A}(B_n) = \mathcal{A}(C_n) &\;:\; x_i - x_j = 0,\ x_i + x_j = 0,\ x_k = 0 \\ \mathcal{A}(D_n) &\;:\; x_i - x_j = 0,\ x_i + x_j = 0. \end{aligned}$$

See Figure 1 for the arrangements $\mathcal{A}(B_2)$ and $\mathcal{A}(D_2)$.

Let us compute the characteristic polynomial $\chi_{\mathcal{A}(B_n)}(q)$. For $p \gg 0$ (actually $p > 2$) and $q = p^m$ we have

$$\chi_{\mathcal{A}(B_n)}(q) = \#\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n \,:\, \alpha_i \neq \pm\alpha_j \ (i \neq j),\ \alpha_i \neq 0 \ (1 \leq i \leq n)\}.$$

Choose $\alpha_1 \in \mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ in $q - 1$ ways. Then choose $\alpha_2 \in \mathbb{F}_q^* - \{\alpha_1, -\alpha_1\}$ in $q - 3$ ways, then α_3 in $q - 5$ ways, etc., to obtain:

$$\chi_{\mathcal{A}(B_n)}(t) = (t-1)(t-3)\cdots(t-(2n-1)).$$

In particular,

$$r(\mathcal{A}(B_n)) = (-1)^n \chi_{\mathcal{A}(B_n)}(-1) = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!.$$

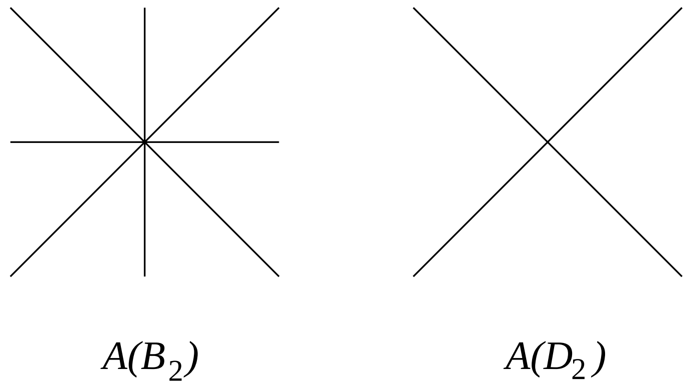


Figure 1. The arrangements $\mathcal{A}(B_2)$ and $\mathcal{A}(D_2)$

By a similar but slightly more complicated argument we get (Exercise 1)

$$\chi_{\mathcal{A}(D_n)}(t) = (t-1)(t-3)\cdots(t-(2n-3))\cdot(t-n+1). \tag{37}$$

Note. Coxeter arrangements are always free in the sense of Theorem 4.14 (a result of Terao [**21**]), but need not be supersolvable. In fact, $\mathcal{A}(A_n)$ and $\mathcal{A}(B_n)$ are supersolvable, but $\mathcal{A}(D_n)$ is not supersolvable for $n \geq 4$ [**3**, Thm. 5.1].

5.2. The Shi arrangement

We next consider a modification (or deformation) of the braid arrangement called the *Shi arrangement* [**15**, §7] and denoted $\mathcal{S}_n$. It consists of the hyperplanes

$$x_i - x_j = 0, 1, \quad 1 \leq i < j \leq n.$$

Thus $\mathcal{S}_n$ has $n(n-1)$ hyperplanes and $\mathrm{rank}(\mathcal{S}_n) = n-1$. Figure 2 shows the Shi arrangement $\mathcal{S}_3$ in $\ker(x_1+x_2+x_3) \cong \mathbb{R}^2$ (i.e., the space $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\}$).

Theorem 5.16. *The characteristic polynomial of $\mathcal{S}_n$ is given by*

$$\chi_{\mathcal{S}_n}(t) = t(t-n)^{n-1}.$$

Proof. Let p be a large prime. By Theorem 5.15 we have

$$\chi_{\mathcal{S}_n}(p) = \#\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{F}_p^n : i < j \Rightarrow \alpha_i \neq \alpha_j \text{ and } \alpha_i \neq \alpha_j + 1\}.$$

Choose a weak ordered partition $\pi = (B_1,\ldots,B_{p-n})$ of $[n]$ into $p-n$ blocks, i.e., $\bigcup B_i = [n]$ and $B_i \cap B_j = \emptyset$ if $i \neq j$, such that $1 \in B_1$. ("Weak" means that we allow $B_i = \emptyset$.) For $2 \leq i \leq n$ there are $p-n$ choices for j such that $i \in B_j$, so $(p-n)^{n-1}$ choices in all. We will illustrate the following argument with the example $p = 11$, $n = 6$, and

$$\pi = (\{1,4\},\{5\},\emptyset,\{2,3,6\},\emptyset). \tag{38}$$

Arrange the elements of $\mathbb{F}_p$ clockwise on a circle. Place $1,2,\ldots,n$ on some n of these points as follows. Place elements of B_1 consecutively (clockwise) in increasing order with 1 placed at some element $\alpha_1 \in \mathbb{F}_p$. Skip a space and place the elements of B_2 consecutively in increasing order. Skip another space and place the elements of B_3 consecutively in increasing order, etc. For our example (38), say $\alpha_1 = 6$.

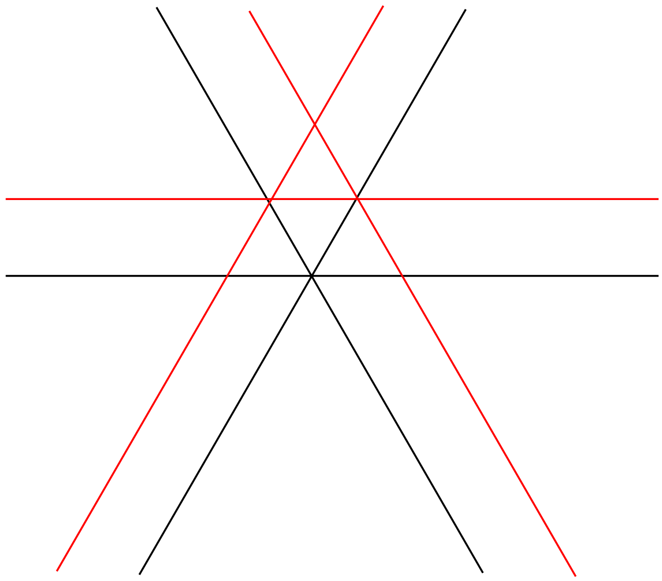
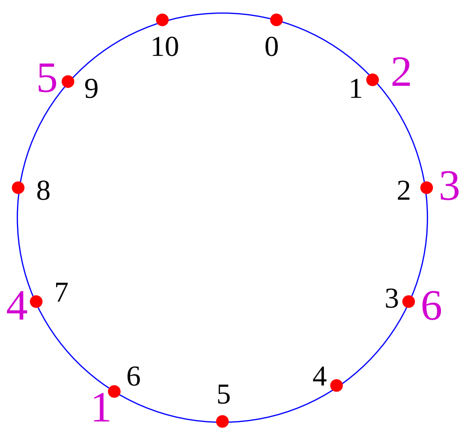


Figure 2. The Shi arrangement $\mathcal{S}_3$ in $\ker(x_1 + x_2 + x_3)$



Let α_i be the position (element of $\mathbb{F}_p$) at which i was placed. For our example we have

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (6, 1, 2, 7, 9, 3).$$

It is easily verified that we have defined a bijection from the $(p-n)^{n-1}$ weak ordered partitions $\pi = (B_1, \ldots, B_{p-n})$ of $[n]$ into $p-n$ blocks such that $1 \in B_1$, together with the choice of $\alpha_1 \in \mathbb{F}_p$, to the set $\mathbb{F}_p^n - \cup_{H \in (\mathcal{S}_n)_p} H$. There are $(p-n)^{n-1}$ choices for π and p choices for α_1, so it follows from Theorem 5.15 that $\chi_{\mathcal{S}_n}(t) = t(t-n)^{n-1}$. $\square$

We obtain the following corollary immediately from Theorem 2.5.

Corollary 5.11. *We have* $r(\mathcal{S}_n) = (n+1)^{n-1}$ *and* $b(\mathcal{S}_n) = (n-1)^{n-1}$.

Note. Since $r(\mathcal{S}_n)$ and $b(\mathcal{S}_n)$ have such simple formulas, it is natural to ask for a direct bijective proof of Corollary 5.11. A number of such proofs are known; a sketch that $r(\mathcal{S}_n) = (n+1)^{n-1}$ is given in Exercise 3.

Note. It can be shown that the cone $c\mathcal{S}_n$ is not supersolvable for $n \geq 3$ (Exercise 4) but is free in the sense of Theorem 4.14.

5.3. Exponential sequences of arrangements

The braid arrangement (in fact, any Coxeter arrangement) is highly symmetrical; indeed, the group of linear transformations that preserves the arrangement acts transitively on the regions. Thus all regions "look the same." The Shi arrangement lacks this symmetry, but it still possesses a kind of "combinatorial symmetry" that allows us to express the characteristic polynomials $\chi_{\mathcal{S}_n}(t)$, for all $n \geq 1$, in terms of the number $r(\mathcal{S}_n)$ of regions.

Definition 5.14. A sequence $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ of arrangements is called an *exponential sequence of arrangements* (ESA) if it satisfies the following three conditions.

1. $\mathcal{A}_n$ is in K^n for some field K (independent of n).
2. Every $H \in \mathcal{A}_n$ is parallel to some hyperplane H' in the braid arrangement $\mathcal{B}_n$ (over K).
3. Let S be a k-element subset of $[n]$, and define

   $\mathcal{A}_n^S = \{H \in \mathcal{A}_n \,:\, H \text{ is parallel to } x_i - x_j = 0 \text{ for some } i, j \in S\}.$

   Then $L(\mathcal{A}_n^S) \cong L(\mathcal{A}_k)$.

Examples of ESA's are given by $\mathcal{A}_n = \mathcal{B}_n$ or $\mathcal{A}_n = \mathcal{S}_n$. In fact, in these cases we have $\mathcal{A}_n^S \cong \mathcal{A}_k \times K^{n-k}$.

The combinatorial properties of ESA's are related to the *exponential formula* in the theory of exponential generating functions [**19**, §5.1], which we now review. Informally, we are dealing with "structures" that can be put on a vertex set V such that each structure is a disjoint union of its "connected components." We obtain a structure on V by partitioning V and placing a connected structure on each block (independently). Examples of such structures are graphs, forests, and posets, but not trees or groups. Let $h(n)$ be the total number of structures on an n-set V (with $h(0) = 1$), and let $f(n)$ be the number that are connected. The exponential formula states that

$$\sum_{n\geq 0} h(n)\frac{x^n}{n!} = \exp \sum_{n\geq 1} f(n)\frac{x^n}{n!}. \tag{39}$$

More precisely, let $f : \mathbb{P} \to R$, where R is a commutative ring. (For our purposes, $R = \mathbb{Z}$ will do.) Define a new function $h : \mathbb{N} \to R$ by $h(0) = 1$ and

$$h(n) = \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} f(\#B_1)f(\#B_2)\cdots f(\#B_k). \tag{40}$$

Then equation (39) holds. A straightforward proof can be given by considering the expansion

$$\begin{aligned}\exp \sum_{n\geq 1} f(n)\frac{x^n}{n!} &= \prod_{n\geq 1} \exp f(n)\frac{x^n}{n!}\\ &= \prod_{n\geq 1}\left(\sum_{k\geq 0} f(n)^k \frac{x^{kn}}{n!^k k!}\right).\end{aligned}$$

We omit the details (Exercise 5).

For any arrangement $\mathcal{A}$ in K^n, define $r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$. Of course if $K = \mathbb{R}$ this coincides with the definition of $r(\mathcal{A})$ as the number of regions of $\mathcal{A}$. We come to the main result concering ESA's.

Theorem 5.17. *Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ be an ESA. Then*

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-t}.$$

Example 5.13. For $\mathfrak{A} = (\mathcal{B}_1, \mathcal{B}_2, \dots)$ Theorem 5.17 asserts that

$$\sum_{n\geq 0} t(t-1)\cdots(t-n+1)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n n!\frac{x^n}{n!}\right)^{-t},$$

as immediately follows from the binomial theorem. On the other hand, if $\mathfrak{A} = (\mathcal{S}_1, \mathcal{S}_2, \dots)$, then we obtain the much less obvious identity

$$\sum_{n\geq 0} t(t-n)^{n-1}\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n(n+1)^{n-1}\frac{x^n}{n!}\right)^{-t}.$$

Proof of Theorem 5.17. By Whitney's theorem (Theorem 2.4) we have for any arrangement $\mathcal{A}$ in K^n that

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \mathcal{B}\text{ central}}} (-1)^{\#\mathcal{B}} t^{n-\operatorname{rank}(\mathcal{B})}.$$

Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$, and let $\mathcal{B} \subseteq \mathcal{A}_n$ for some n. Define $\pi(\mathcal{B}) \in \Pi_n$ to have blocks that are the vertex sets of the connected components of the graph G on $[n]$ with edges

$$E(G) = \{ij : \exists\, x_i - x_j = c \text{ in } \mathcal{B}\}. \tag{41}$$

Define

$$\tilde{\chi}_{\mathcal{A}_n}(t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \mathcal{B}\text{ central}\\ \pi(\mathcal{B})=[n]}} (-1)^{\#\mathcal{B}} t^{n-\operatorname{rk}(\mathcal{B})}.$$

Then

$$\begin{aligned}
\chi_{\mathcal{A}_n}(t) &= \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \mathcal{B}\text{ central}\\ \pi(\mathcal{B})=\pi}} (-1)^{\#\mathcal{B}} t^{n-\operatorname{rk}(\mathcal{B})}\\
&= \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} \tilde{\chi}_{\mathcal{A}_{\#B_1}}(t)\tilde{\chi}_{\mathcal{A}_{\#B_2}}(t)\cdots\tilde{\chi}_{\mathcal{A}_{\#B_k}}(t).
\end{aligned}$$

Thus by the exponential formula (39),

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \exp\sum_{n\geq 1}\tilde{\chi}_{\mathcal{A}_n}(t)\frac{x^n}{n!}.$$

But $\pi(\mathcal{B}) = [n]$ if and only if $\mathrm{rk}(\mathcal{B}) = n-1$, so $\tilde{\chi}_{\mathcal{A}_n}(t) = c_n t$ for some $c_n \in \mathbb{Z}$. We therefore get

$$\begin{aligned} \sum_{n\geq 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} &= \exp t \sum_{n\geq 1} c_n \frac{x^n}{n!} \\ &= \left(\sum_{n\geq 0} b_n \frac{x^n}{n!}\right)^t, \end{aligned}$$

where $\exp \sum_{n\geq 1} c_n \frac{x^n}{n!} = \sum_{n\geq 0} b_n \frac{x^n}{n!}$. Put $t=-1$ to get

$$\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!} = \left(\sum_{n\geq 0} b_n \frac{x^n}{n!}\right)^{-1},$$

from which it follows that

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-t}.$$

$\square$

For a generalization of Theorem 5.17, see Exercise 10.

5.4. The Catalan arrangement

Define the *Catalan arrangement* $\mathcal{C}_n$ in K^n, where $\mathrm{char}(K) \neq 2$, by

$$Q_{\mathcal{C}_n}(x) = \prod_{1\leq i<j\leq n} (x_i - x_j)(x_i - x_j - 1)(x_i - x_j + 1).$$

Equivalently, the hyperplanes of $\mathcal{C}_n$ are given by

$$x_i - x_j = -1, 0, 1, \quad 1 \leq i < j \leq n.$$

Thus $\mathcal{C}_n$ has $3\binom{n}{2}$ hyperplanes, and $\mathrm{rank}(\mathcal{C}_n) = n-1$.

Assume now that $K = \mathbb{R}$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting coordinates, i.e.,

$$w \cdot (x_1, \dots, x_n) = (x_{w(1)}, \dots, x_{w(n)}).$$

Here we are multiplying permutations left-to-right, e.g., $(1,2)(2,3) = (1,3,2)$ (in cycle form), so $vw \cdot \alpha = v \cdot (w \cdot \alpha)$. Both $\mathcal{B}_n$ and $\mathcal{C}_n$ are $\mathfrak{S}_n$-invariant, i.e., $\mathfrak{S}_n$ permutes the hyperplanes of these arrangements. Hence $\mathfrak{S}_n$ also permutes their regions, and each region $x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)}$ of $\mathcal{B}_n$ is divided "in the same way" in $\mathcal{C}_n$. In particular, if $r_0(\mathcal{C}_n)$ denotes the number of regions of $\mathcal{C}_n$ contained in some fixed region of $\mathcal{B}_n$, then $r(\mathcal{C}_n) = n! r_0(\mathcal{C}_n)$. See Figure 3 for $\mathcal{C}_3$ in the ambient space $\ker(x_1 + x_2 + x_3)$, where the hyperplanes of $\mathcal{B}_3$ are drawn as solid lines and the remaining hyperplanes as dashed lines. Each region of $\mathcal{B}_3$ contains five regions of $\mathcal{C}_3$, so $r(\mathcal{C}_3) = 6 \cdot 5 = 30$.

We can compute $r(\mathcal{C}_n)$ (or equivalently $r_0(\mathcal{C}_n)$) by a direct combinatorial argument. Let R_0 denote the region $x_1 > x_2 > \cdots > x_n$ of $\mathcal{B}_n$. The regions of $\mathcal{C}_n$ contained in R_0 are determined by those $i < j$ such that $x_i - x_j < 1$. We need only specify the *maximal* intervals $[i,j]$ such that $x_i - x_j < 1$, i.e., if $a \leq i < j \leq b$ and $x_a - x_b < 1$, then $a = i$ and $b = j$. It is easy to see that any such specification of maximal intervals determines a region of $\mathcal{C}_n$ contained in R_0. Thus $r_0(\mathcal{C}_n)$ is equal

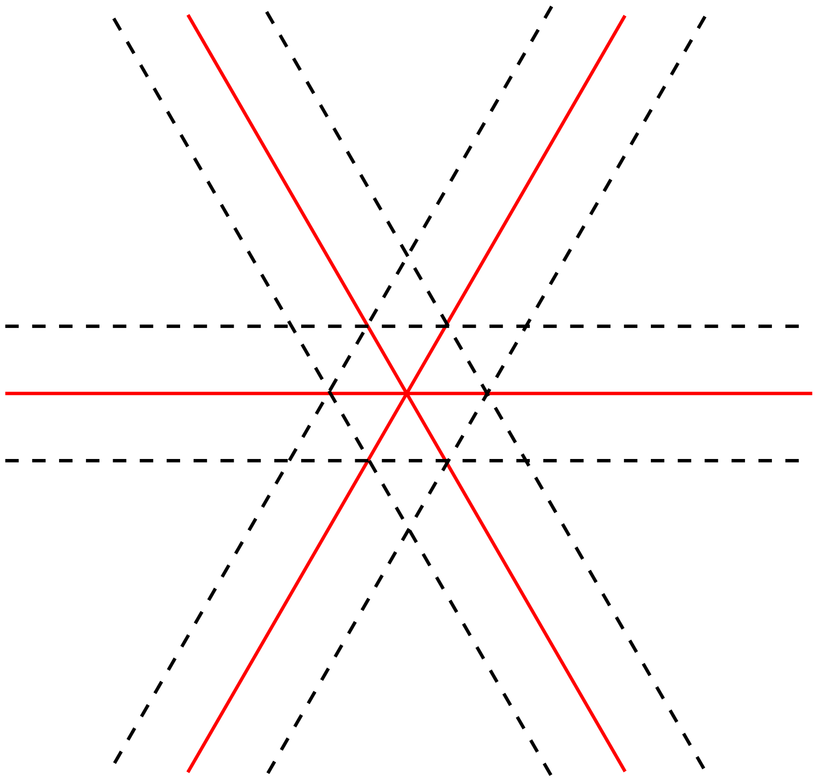


Figure 3. The Catalan arrangement $\mathcal{C}_3$ in $\ker(x_1 + x_2 + x_3)$

to the number of *antichains* A of strict intervals of $[n]$, i.e., sets A of intervals $[i, j]$, where $1 \leq i < j \leq n$, such that no interval in A is contained in another. ("Strict" means that $i = j$ is not allowed.) It is known (equivalent to [**19**, Exer. 6.19(bbb)]) that the number of such antichains is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. For the sake of completeness we give a bijection between these antichains and a standard combinatorial structure counted by Catalan numbers, viz., lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$, never rising above the line $y = x$ ([**19**, Exer. 6.19(h)]). Given an antichain A of intervals of $[n]$, there is a unique lattice path of the claimed type whose "outer corners" (a step $(1, 0)$ followed by $(0, 1)$) consist of the points $(j, i - 1)$ where $[i, j] \in A$, together with the points $(i, i - 1)$ where no interval in A contains i. Figure 4 illustrates this bijection for $n = 8$ and $A = \{[1, 4], [3, 5], [7, 8]\}$.

We have therefore proved the following result. For a refinement, see Exercise 11.

Proposition 5.14. *The number of regions of the Catalan arrangement* $\mathcal{C}_n$ *is given by* $r(\mathcal{C}_n) = n!C_n$. *Each region of* $\mathcal{B}_n$ *contains* C_n *regions of* $\mathcal{C}_n$.

In fact, there is a simple formula for the characteristic polynomial $\chi_{\mathcal{C}_n}(t)$.

Theorem 5.18. *We have*

$$\chi_{\mathcal{C}_n}(t) = t(t - n - 1)(t - n - 2)(t - n - 3) \cdots (t - 2n + 1).$$

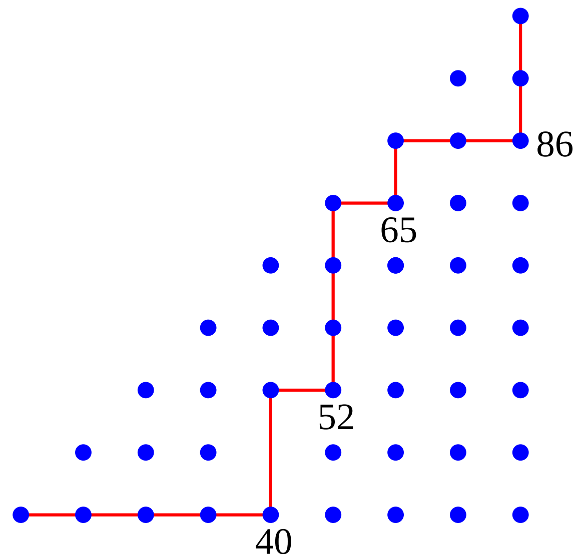


Figure 4. A bijection corresponding to $A = \{[1,4],[3,5],[7,8]\}$

Proof. Clearly the sequence $(\mathcal{C}_1, \mathcal{C}_2, \dots)$ is an ESA, so by Theorem 5.17 we have

$$\begin{aligned}\sum_{n\geq 0} \chi_{\mathcal{C}_n}(t)\frac{x^n}{n!} &= \left(\sum_{n\geq 0}(-1)^n n! C_n \frac{x^n}{n!}\right)^{-t} \\ &= \left(\sum_{n\geq 0}(-1)^n C_n x^n\right)^{-t}.\end{aligned}$$

One method for expanding this series is to use the *Lagrange inversion formula* [**19**, Thm. 5.4.2]. Let $F(x) = a_1 x + a_2 x^2 + \cdots$ be a formal power series over K, where $\mathrm{char}(K) = 0$ and $a_1 \neq 0$. Then there exists a unique formal power series $F^{\langle -1\rangle} = a_1^{-1}x + \cdots$ satisfying

$$F(F^{\langle -1\rangle}(x)) = F^{\langle -1\rangle}(F(x)) = x.$$

Let $k, t \in \mathbb{Z}$. The Lagrange inversion formula states that

$$t[x^t]F^{\langle -1\rangle}(x)^k = k[x^{t-k}]\left(\frac{x}{F(x)}\right)^t. \tag{42}$$

Let $y = \sum_{n\geq 0}(-1)^n C_n x^{n+1}$. By a fundamental property of Catalan numbers, $y^2 = -y + x$. Hence $y = (x + x^2)^{\langle -1\rangle}$. Substitute $t - n$ for k and apply equation (42) to $y = F(x)$, so $F^{\langle -1\rangle}(x) = x + x^2$:

$$t[x^t](x + x^2)^{t-n} = (t-n)[x^n]\left(\frac{x}{y}\right)^t. \tag{43}$$

The right-hand side of (43) is just

$$(t-n)[x^n]\left(\frac{y}{x}\right)^{-t} = \frac{(t-n)\chi_{\mathcal{C}_n}(t)}{n!}.$$

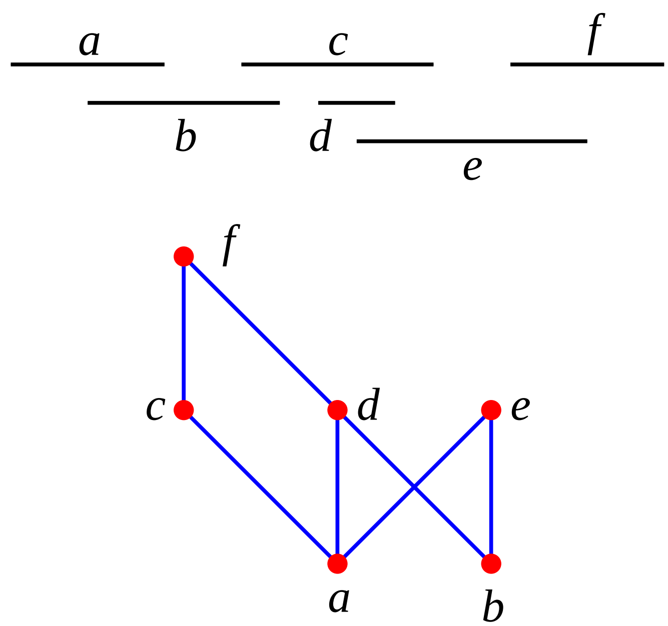


Figure 5. An example of an interval order

The left-hand side of (43) is given by

$$t[x^t]x^{t-n}(1+x)^{t-n} = t\binom{t-n}{n} = \frac{t(t-n)(t-n-1)\cdots(t-2n+1)}{n!}.$$

It follows that

$$\chi_{\mathcal{C}_n}(t) = t(t-n-1)(t-n-2)(t-n-3)\cdots(t-2n+1)$$

for all $t \in \mathbb{Z}$. It then follows easily (e.g., using the fact that a polynomial in one variable over a field of characteristic 0 is determined by its values on $\mathbb{Z}$) that this equation holds when t is an indeterminate. $\square$

Note. It is not difficult to give an alternative proof of Theorem 5.18 based on the finite field method (Exercise 12).

5.5. Interval orders

The subject of interval orders has a long history (see [**10**][**23**]), but only recently [**20**] was their connection with arrangements noticed. Let $P = \{I_1, \ldots, I_n\}$ be a finite set of closed intervals $I_i = [a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$. Partially order P by defining $I_i < I_j$ if $b_i < a_j$, i.e., I_i lies entirely to the left of I_j on the real number line. A poset isomorphic to P is called an *interval order*. Figure 5 gives an example of six intervals and the corresponding interval order. It is understood that the real line lies below and parallel to the line segments labelled $a, \ldots, f$, and that the actual intervals are the projections of these line segments to $\mathbb{R}$. If all the intervals I_i have length one, then P is called a *semiorder* or *unit interval order*.

We will be considering both labelled and unlabelled interval orders. A *labelled* interval order is the same as an interval order on a set S, often taken to be $[n]$. If an interval order P corresponds to intervals $I_1, \ldots, I_n$, then there is a natural

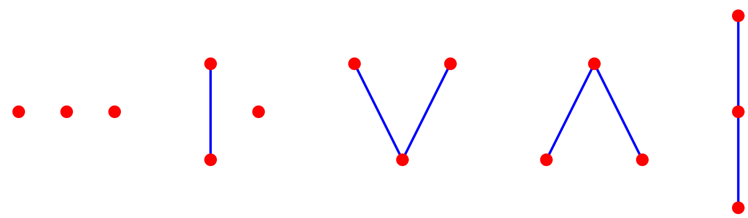


Figure 6. The number of labelings of semiorders with three elements

labeling of P, viz., label the element corresponding to I_i by i. Thus the intervals $I_1 = [0,1]$ and $I_2 = [2,3]$ correspond to the labelled interval order P_1 defined by $1 < 2$, while the intervals $I_1 = [2,3]$ and $I_2 = [0,1]$ correspond to P_2 defined by $2 < 1$. Note that P_1 and P_2 are different labelled interval orders but are isomorphic as posets. As another example, consider the intervals $I_1 = [0,2]$ and $I_2 = [1,3]$. The corresponding labelled interval order P consists of the disjoint points 1 and 2. If we now let $I_1 = [1,3]$ and $I_2 = [0,2]$, then we obtain the *same* labelled interval order (or labelled poset) P, although the intervals themselves have been exchanged. An *unlabelled* interval order may be regarded as an isomorphism class of interval orders; two intervals orders P_1 and P_2 represent the same unlabelled interval order if and only if they are isomorphic. Of course our discussion of labelled and unlabelled interval orders applies equally well to semiorders.

Figure 6 shows the five nonisomorphic (or unlabelled) interval orders (which for three vertices coincides with semiorders) with three vertices, and below them the number of distinct labelings. (In general, the number of labelings of an n-element poset P is $n!/\#\mathrm{Aut}(P)$, where $\mathrm{Aut}(P)$ denotes the automorphism group of P.) It follows that there are 19 labelled interval orders or labelled semiorders on a 3-element set.

The following proposition collects some basic results on interval orders. We simply state them without proof. Only part (a) is needed in what follows (Lemma 5.6). We use the notation $\boldsymbol{i}$ to denote an i-element chain and $P+Q$ to denote the disjoint union of the posets P and Q.

Proposition 5.15. (a) *A finite poset is an interval order if and only if it has no induced subposet isomorphic to* $\mathbf{2}+\mathbf{2}$.

(b) *A finite poset is a semiorder if and only if it has no induced subposet isomorphic to* $\mathbf{2}+\mathbf{2}$ *or* $\mathbf{3}+\mathbf{1}$.

(c) *A finite poset P is a semiorder if and only if its elements can be ordered as $I_1, \dots, I_n$ so that the incidence matrix of P (i.e., the matrix $M = (m_{ij})$, where $m_{ij} = 1$ if $I_i < I_j$ and $m_{ij} = 0$ otherwise) has the form shown below. Moreover, all such semiorders are nonisomorphic.*

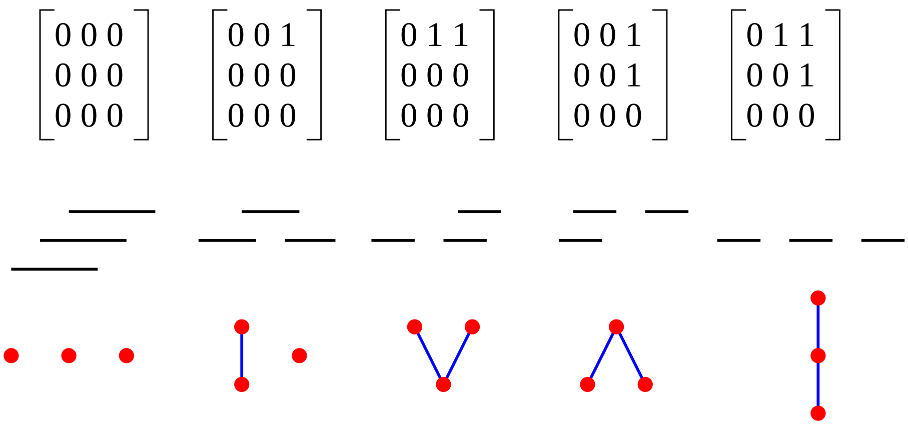
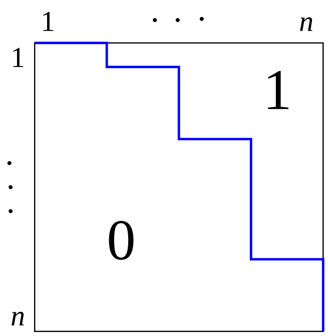


Figure 7. The semiorders with three elements



In (c) above, the southwest boundary of the positions of the 1's in M form a lattice path which by suitable indexing goes from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$, never rising above $y = x$. Since the number of such lattice paths is the Catalan number C_n, it follows that the number of nonisomorphic n-element semiorders is C_n. Later (Proposition 5.17) we will give a proof based on properties of a certain arrangement. Figure 7 illustrates Proposition 5.15(c) when $n = 3$. It shows the matrices M, the corresponding set of unit intervals, and the associated semiorder.

Let $\ell_1, \ldots, \ell_n > 0$ and set $\eta = (\ell_1, \ldots, \ell_n)$. Let $\mathcal{P}_\eta$ denote the set of all interval orders P on $[n]$ such that there exist a set $I_1, \ldots, I_n$ of intervals corresponding to P (with I_i corresponding to $i \in P$) such that $\ell(I_i) = \ell_i$. In other words, $i \overset{P}{<} j$ if and only if I_i lies entirely to the left of I_j. For instance, it follows from Figure 6 that $\#\mathcal{P}_{(1,1,1)} = 19$.

We now come to the connection with arrangements. Given $\eta = (\ell_1, \ldots, \ell_n)$ as above, define the arrangement $\mathcal{I}_\eta$ in $\mathbb{R}^n$ by letting its hyperplanes be given by

$$x_i - x_j = \ell_i, \quad i \neq j.$$

(Note the condition $i \neq j$, not $i < j$.) Thus $\mathcal{I}_\eta$ has rank $n-1$ and $n(n-1)$ hyperplanes (since $\ell_i > 0$). Figure 8 shows the arrangement $\mathcal{I}_{(1,1,1)}$ in the space $\ker(x_1 + x_2 + x_3)$.

Proposition 5.16. *Let $\eta \in \mathbb{R}^n_+$. Then $r(\mathcal{I}_\eta) = \#\mathcal{P}_\eta$.*

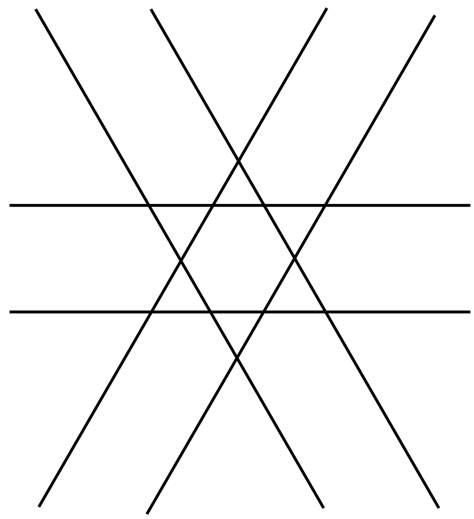


Figure 8. The arrangement $\mathcal{I}_{(1,1,1)}$ in the space $\ker(x_1 + x_2 + x_3)$

Proof. Let $(x_1, \ldots, x_n)$ belong to some region R of $\mathcal{I}_\eta$. Define the interval $I_i = [x_i - \ell_i, x_i]$. The region R is determined by whether $x_i - x_j < \ell_i$ or $x_i - x_j > \ell_i$. Equivalently, $I_i \not> I_j$ or $I_i > I_j$ in the ordering on intervals that defines interval orders. Hence the number of possible interval orders corresponding to intervals $I_1, \ldots, I_n$ with $\ell(I_i) = \ell_i$ is just $r(\mathcal{I}_\eta)$. $\square$

Consider the case $\ell_1 = \cdots = \ell_n = 1$, so we are looking at the *semiorder arrangement* $x_i - x_j = 1$ for $i \neq j$. We abbreviate $(1, 1, \ldots, 1)$ as 1^n and denote this arrangement by $\mathcal{I}_{1^n}$. By the proof of Proposition 5.16 the regions of $\mathcal{I}_{1^n}$ are in a natural bijection with semiorders on $[n]$.

Now note that $\mathcal{C}_n = \mathcal{I}_{1^n} \cup \mathcal{B}_n$, where $\mathcal{C}_n$ denotes the Catalan arrangement. Fix a region R of $\mathcal{B}_n$, say $x_1 < x_2 < \cdots < x_n$. Then the number of regions of $\mathcal{I}_{1^n}$ that intersect R is the number of semiorders on $[n]$ that correspond to (unit) intervals $I_1, \ldots, I_n$ with right endpoints $x_1 < x_2 < \cdots < x_n$. Another set $I'_1, \ldots, I'_n$ of unit intervals $I'_i = [x'_i - 1, x'_i]$ with $x'_1 < x'_2 < \cdots < x'_n$ defines a different region from that defined by $I_1, \ldots, I_n$ if and only if the corresponding semiorders are *nonisomorphic*. It follows that the number of nonisomorphic semiorders on $[n]$ is equal to the number of regions of $\mathcal{I}_{1^n}$ intersecting the region $x_1 < x_2 < \cdots < x_n$ of $\mathcal{B}_n$. Since $\mathcal{C}_n = \mathcal{I}_{1^n} \cup \mathcal{B}_n$, there follows from Proposition 5.14 the following result of Wine and Freunde [**24**].

Proposition 5.17. *The number $u(n)$ of nonisomorphic n-element semiorders is given by*

$$u(n) = \frac{1}{n!} r(\mathcal{C}_n) = C_n.$$

Figure 9 shows the nonisomorphic 3-element semiorders corresponding to the regions of $\mathcal{C}_n$ intersecting the region $x_1 < x_2 < \cdots < x_n$ of $\mathcal{B}_n$.

We now come to the problem of determining $r(\mathcal{I}_{1^n})$, the number of semiorders on $[n]$.

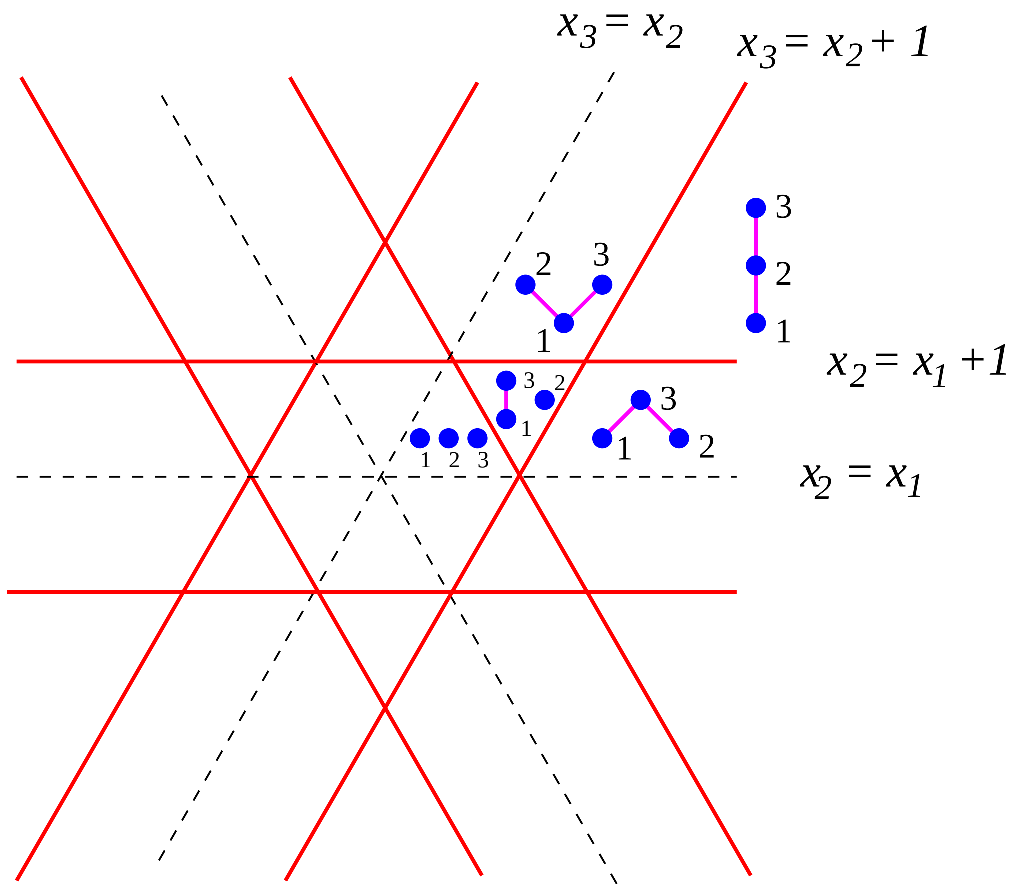


Figure 9. The nonisomorphic 3-element semiorders as regions of $\mathcal{C}_{1^n}$

Theorem 5.19. *Fix distinct real numbers $a_1, a_2, \ldots, a_m > 0$. Let $\mathcal{A}_n$ be the arrangement in $\mathbb{R}^n$ with hyperplanes*

$$\mathcal{A}_n : \quad x_i - x_j = a_1, \ldots, a_m, \quad i \neq j,$$

and let $\mathcal{A}_n^* = \mathcal{A}_n \cup \mathcal{B}_n$. Define

$$\begin{aligned} F(x) &= \sum_{n \geq 1} r(\mathcal{A}_n) \frac{x^n}{n!} \\ G(x) &= \sum_{n \geq 1} r(\mathcal{A}_n^*) \frac{x^n}{n!}. \end{aligned}$$

Then $F(x) = G(1 - e^{-x})$.

Proof. Let $c(n, k)$ denote the number of permutations w of n objects with k cycles (in the disjoint cycle decomposition of w). The integer $c(n, k)$ is known as a *signless Stirling number of the first kind* and for fixed k has the exponential generating function

$$\sum_{n\geq 0} c(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\log(1-x)^{-1}\right)^k. \tag{44}$$

For futher information, see e.g. [**18**, pp. 17–20][**19**, (5.25)].

We have

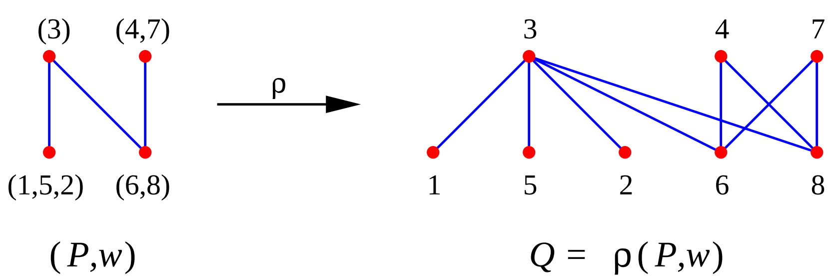
$$\begin{aligned} F(x) = G(1-e^{-x}) \Leftrightarrow G(x) &= F(\log(1-x)^{-1}) \\ &= \sum_{k\geq 1} r(\mathcal{A}_k)\frac{1}{k!}\left(\log(1-x)^{-1}\right)^k \\ &= \sum_{k\geq 1} r(\mathcal{A}_k)\sum_{n\geq 0} c(n,k)\frac{x^n}{n!}. \end{aligned}$$

It follows that we need to show that

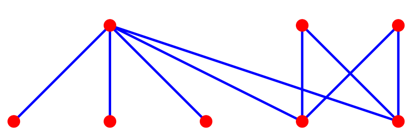
$$r(\mathcal{A}_n^*) = \sum_{k=1}^{n} c(n,k) r(\mathcal{A}_k). \tag{45}$$

For simplicity we consider only the case $m=1$ and $a_1=1$, but the argument is completely analogous in the general case. When $m=1$ and $a_1=1$ we have that $r(\mathcal{A}_n^*) = n!C_n$ and that $r(\mathcal{A}_n)$ is the number of semiorders on $[n]$. Thus it suffices to give a map $(P,w) \overset{\rho}{\mapsto} Q$, where $w \in \mathfrak{S}_k$ and P is a semiorder whose elements are labelled by the cycles of w, and where Q is an unlabelled n-element semiorder, such that ρ is $n!$-to-1, i.e., every Q appears exactly $n!$ times as an image of some (P,w).

Choose $w \in \mathfrak{S}_n$ with k cycles in $c(n,k)$ ways, and make these cycles the vertices of a semiorder P in $r(\mathcal{A}_k)$ ways. Define a new poset $\rho(P,w)$ as follows: if the cycle $(c_1,\ldots,c_j)$ is an element of P, then replace it with an antichain with elements $c_1,\ldots,c_j$. Given $1 \leq c \leq n$, let $C(c)$ be the cycle of w containing c. Define $c < d$ in $\rho(P,w)$ if $C(c) < C(d)$ in P. We illustrate this definition with $n=8$ and $w=(1,5,2)(3)(6,8)(4,7)$:



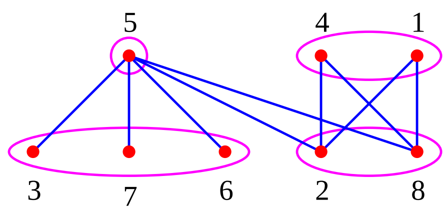
Given an unlabelled n-element semiorder Q, such as



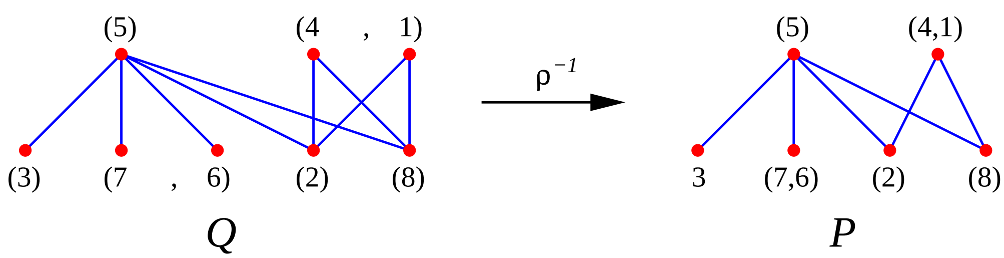
we now show that there are exactly $n!$ pairs (P, w) for which $\rho(P, w) \cong Q$. Call a pair of elements $x, y \in Q$ *autonomous* if for all $z \in Q$ we have

$$x < z \Leftrightarrow y < z, \qquad x > z \Leftrightarrow y > z.$$

Equivalently, the map $\tau : Q \to Q$ transposing x, y and fixing all other $z \in Q$ is an automorphism of Q. Clearly the relation of being autonomous is an equivalence relation. Partition Q into its autonomous equivalence classes. Regard the elements of Q as being distinguished, and choose a bijection (labeling) $\varphi : Q \to [n]$ (in $n!$ ways). Fix a linear ordering (independent of φ) of the elements in each equivalence class. (The linear ordering of the elements in each equivalence class in the diagram below is left-to-right.)



In each class, place a left parenthesis before each left-to-right maximum, and place a right parenthesis before each left parenthesis and at the end. (This is the bijection $\mathfrak{S}_n \to \mathfrak{S}_n$, $\hat{w} \mapsto w$, in [**18**, p. 17].) Merge the elements $c_1, c_2, \ldots, c_j$ (appearing in that order) between each pair of parentheses into a single element labelled with the cycle $(c_1, c_2, \ldots, c_j)$.



We have thus obtained a poset P whose elements are labelled by the cycles of a permutation $w \in \mathfrak{S}_n$, such that $\rho(P, w) = Q$. For each unlabelled Q, there are exactly $n!$ pairs (P, w) (where the poset P is labelled by the cycles of $w \in \mathfrak{S}_n$) for which $\rho(P, w) \cong Q$. Since by Proposition 5.17 there are C_n nonisomorphic n-element semiorders, we get

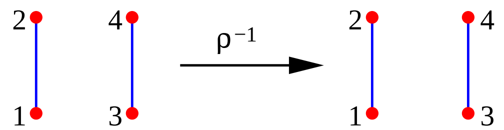
$$n! C_n = \sum_{k=1}^{n} c(n, k) r(\mathcal{A}_k).$$

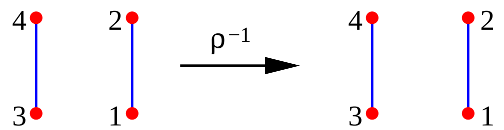
$\square$

Note. Theorem 5.19 can also be proved using Burnside's lemma (also called the Cauchy-Frobenius lemma) from group theory.

To test one's understanding of the proof of Theorem 5.19, consider why it doesn't work for *all* posets. In other words, let $f(n)$ denote the number of posets on $[n]$ and $g(n)$ the number of nonisomorphic n-element posets. Set $F(x) = \sum f(n) \frac{x^n}{n!}$ and $G(x) = \sum g(n) x^n$. Why doesn't the above argument show that $G(x) = F(1 -$

e^{-x})? Let $Q = \mathbf{2} + \mathbf{2}$ (the unique obstruction to being an interval order, by Proposition 5.15(a)). The autonomous classes have one element each. Consider the two labelings $\varphi : Q \to [4]$ and the corresponding ρ^{-1}:





We obtain the same labelled posets in both cases, so the proof of Theorem 5.19 fails. The key property of interval orders that the proof of Theorem 5.19 uses implicitly is the following.

Lemma 5.6. *If $\sigma : P \to P$ is an automorphism of the interval order P and $\sigma(x) = \sigma(y)$, then x and y are autonomous.*

Proof. Assume not. Then there exists an element $s \in P$ satisfying $s > x$, $s \not> y$ (or dually). Since $\sigma(x) = y$, there must exist $t \in P$ satisfying $t > y$, $t \not> x$. But then $\{x, s, y, t\}$ form an induced $\mathbf{2} + \mathbf{2}$, so by Proposition 5.15(a) P is not an interval order. $\square$

Specializing $m = 1$ and $a_1 = 1$ in Theorem 5.19 yields the following corollary, due first (in an equivalent form) to Chandon, Lemaire and Pouget [**8**].

Corollary 5.12. *Let $f(n)$ denote the number of semiorders on $[n]$ (or n-element labelled semiorders). Then*

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = C(1 - e^{-x}),$$

where

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

5.6. Intervals with generic lengths

A particularly interesting class of interval orders are those corresponding to intervals with specified *generic lengths* $\eta = (\ell_1, \dots, \ell_n)$. Intuitively, this means that the intersection poset $P(\mathcal{I}_\eta)$ is as "large as possible." One way to make this precise is to say that η is *generic* if $P(\mathcal{I}_\eta) \cong P(\mathcal{I}_{\eta'})$, where $\eta' = (\ell'_1, \dots, \ell'_n)$ and the ℓ'_i's are linearly independent over $\mathbb{Q}$. Thus if η is generic, then the intersection poset $L(\mathcal{I}_\eta)$ does not depend on η, but rather only on n. In particular, $r(\mathcal{I}_\eta)$ does not depend on η (always assuming η is generic). Hence by Proposition 5.16, the number $\#\mathcal{P}_\eta$ of labelled interval orders corresponding to intervals $I_1, \dots, I_n$ with $\ell(I_i) = \ell_i$ depends only on n. This fact is not at all obvious combinatorially, since the interval orders themselves do depend on η. For instance, it is easy to see that $\eta = (1, 1.0001, 1.001, 1.01, 1.1)$ is generic and that no corresponding interval order can be isomorphic to $\mathbf{4}+\mathbf{1}$. On the other hand, $\eta = (1, 10, 100, 1000, 10000)$ is also generic, but this time there is a corresponding interval order isomorphic to $\mathbf{4}+\mathbf{1}$. (See Exercise 17.)

The preceding discussion raises the question of computing $\#\mathcal{P}_n$ when η is generic. We write $\mathcal{G}_n$ for the corresponding interval order $x_i - x_j = \ell_i$, $i \neq j$, since the intersection poset depends only on n. The following result is a nice application of arrangements to "pure" enumeration; no proof is known except the one sketched here.

Theorem 5.20. *Let*

$$z = \sum_{n\geq 0} r(\mathcal{G}_n)\frac{x^n}{n!} = 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + 2831\frac{x^5}{5!} + \cdots .$$

Define a power series

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + 631\frac{x^4}{4} + \cdots$$

by $1 = y(2 - e^{xy})$. *Equivalently,*

$$y = 1 + \left(\frac{1}{1+x}\log\frac{1+2x}{1+x}\right)^{\langle -1\rangle}.$$

Then z *is the unique power series satisfying* $z'/z = y^2$, $z(0) = 1$.

Note. The condition $z'/z = y^2$ can be rewritten as $z = \exp\int y^2\,dx$.

Sketch of proof. Putting $t = -1$ in Theorem 2.4 gives

$$r(\mathcal{G}_n) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{G}_n\\ \mathcal{B}\text{ central}}} (-1)^{\#\mathcal{B}-\mathrm{rk}(\mathcal{B})}. \tag{46}$$

Given a central subarrangement $\mathcal{B} \subseteq \mathcal{G}_n$, define a digraph (directed graph) $G_{\mathcal{B}}$ on $[n]$ by letting $i \to j$ be a (directed) edge if the hyperplane $x_i - x_j = \ell_i$ belongs to $\mathcal{B}$. One then shows that as an undirected graph $G_{\mathcal{B}}$ is bipartite, i.e., the vertices can be partitioned into two subsets U and V such that all edges connect a vertex in U to a vertex in V. The pair (U, V) is called a *vertex bipartition* of $G_{\mathcal{B}}$. Moreover, if B is a block of $G_{\mathcal{B}}$ (as defined preceding Proposition 4.11), say with vertex bipartition (U_B, V_B), then either all edges of B are directed from U_B to V_B, or all edges are directed from V_B to U_B. It can also be seen that all such directed bipartite graphs can arise in this way. It follows that equation (46) can be rewritten

$$r(\mathcal{G}_n) = (-1)^n \sum_G (-1)^{e(G)+c(G)} 2^{b(G)}, \tag{47}$$

where G ranges over all (undirected) bipartite graphs on $[n]$, $e(G)$ denotes the number of edges of G, and $b(G)$ denotes the number of blocks of G.

Equation (47) reduces the problem of determining $r(\mathcal{G})$ to a (rather difficult) problem in enumeration, whose solution may be found in [**14**, §6]. ☐

5.7. Other examples

There are two additional arrangements related to the braid arrangement that involve nice enumerative combinatorics. We merely repeat the definitions here from Lecture 1 and assemble some of their basic properties in Exercises 19–28.

The *Linial arrangement* in K^n is given by the hyperplanes $x_i - x_j = 1$, $1 \le i < j \le n$. It consists of "half" of the semiorder arrangement $\mathcal{I}_{1^n}$. Despite its similarity to $\mathcal{I}_{1^n}$, it is considerably more difficult to obtain its characteristic polynomial and other enumerative invariants. Finally the *threshold arrangement* in K^n is given by the hyperplanes $x_i + x_j = 0$, $1 \le i < j \le n$. It is a subarrangement of the Coxeter arrangements $\mathcal{A}(B_n)$ $(=\mathcal{A}(C_n))$ and $\mathcal{A}(D_n)$.