Pak
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More on Young tableaux

① RSK $S_n \to \{(A,B) \mid A,B \in SYT(\lambda),\ \lambda \vdash n\}$

cor $n! = \sum_{\lambda \vdash n} |SYT(\lambda)|^2$

New bijection: jeu-de-taquin (Aunt Cathy's sliding # tile game)

Schützenberger's bij $S_n \leftrightarrow \{(A,B)\}$

$\varphi$: take $\sigma \in S_n$, write from bottom left to top right as long diagonal, then slide everything up as much as can legally (+ still satisfy SYT ineq) then left as much as possible, then fill in corner hole by... um... example unclear

Actual bijection is $\alpha$ sending $\sigma$ to $(\varphi(\sigma), \varphi(\sigma^{-1}))$

Theorem: $\alpha$ is a bijection between $S_n$ + what we want it to be

Now let's go back to our earlier fascinating result

$$|SYT(\lambda)| = \frac{n!}{\prod_{x \in \lambda} h_x}$$

'90s: $|SYT(\lambda)| \cdot \prod h_x = n!$

Let's try to make a bijection

$S_n \leftrightarrow (A, \{f_x, x \in \lambda\})$ where $A \in SYT(\lambda)$ and $f_x$ is some number to be explained later

Reminder: bubble sort

Now he's ready to define 2-d bubble sorting

$\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}$

Put numbers in $\lambda$ willy-nilly. Bubble in order bottom $\to$ top column by column, right to left, bubble w.r.t. $b$ if $a > c > b$, $c$ if $a > b > c$

Claim: 2-dim bubble sort is an equinumerous surjection $S_n \to SYT(\lambda)$, i.e. surjection $\forall A, A' \in SYT(\lambda)$ $|\text{pre-im } A| = |\text{pre-im } A'|$

For $\boxed{\ \ \ \ \ \ \ \ \ \ }$, let $a_i$ = # steps $\sigma(i)$ makes to the right during bubble
This gives bij between permutations + inv. index ("obvious")

$\begin{matrix} x & \bullet \\ \bullet & x \end{matrix}$ $\Rightarrow x \to x+1$

$\begin{matrix} x \\ y \end{matrix} \to \begin{matrix} y \\ x \end{matrix}$ $\Rightarrow \binom{x}{y} \to \binom{y-1}{x}$

Analogously, for each position, start w/ all 0, if entry in that position's right neighbor swaps w/ it in $A$, then in this labelling add 1. If entry swaps w/ one below, swap labelling + subtract 1 from it

So each box has range of size its hook length, going from $-$height to $+$ length