Crossing Numbers

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Abstract — For any simple graph $G = (V, E)$, we can define four types of crossing number: crossing number, rectilinear crossing number, odd-crossing number, and pairwise crossing number. We discuss the relationship of them.

1 INTRODUCTION

The concept "drawing" is defined in a variety of distinct ways in the literature. In this paper, we use the definition in the paper by Pach and Tóth [1].

A *drawing* D of a simple graph G is a mapping f of the vertices and edges of G to the plane, assigning to each vertex a distinct point in the plane and to each edge uv a continuous arc (i.e. a homeomorphic image of a closed interval), which is called an *edge* of the drawing D, connecting $f(u)$ and $f(v)$, and satisfying

1. A edge of D doesn't pass through the image of a vertex other than its endpoints.
2. Two edges of D have a finite number of intersection points.
3. Any intersection of two edges of D is a proper crossing.
4. No three edges of D have a common intersection point.

In such a drawing, the intersection of two edges is called a *crossing* (a common endpoint of two edges does not count as a crossing). Now we can define the four types of crossing number.

DEFINITION. Let G be a simple graph.

1. The *crossing number* of G, $\mathrm{cr}(G)$, is the minimum number of crossings over all drawing of G.

2. The *rectilinear crossing number* of G, $\mathrm{lin}-\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G, where every edge is represented by a line segment.

3. The *odd-crossing number* of G, $\mathrm{odd}-\mathrm{cr}(G)$, is the minimum number of pairs of edges with odd number of crossings over all drawings of G.

4. The *pairwise crossing number* of G, $\mathrm{pair}-\mathrm{cr}(G)$, is the minimum number of pairs of crossing edges over all drawing of G.

Clearly, we have the following inequality.

THEOREM 1. For any simple graph G, we have

$$\mathrm{odd}-\mathrm{cr}(G) \leq \mathrm{pair}-\mathrm{cr}(G) \leq \mathrm{cr}(G) \leq \mathrm{lin}-\mathrm{cr}(G).$$

PROOF. It's trivial that $\mathrm{odd}-\mathrm{cr}(G) \leq \mathrm{pair}-\mathrm{cr}(G)$ and $\mathrm{cr}(G) \leq \mathrm{lin}-\mathrm{cr}(G)$. It's enough to show that $\mathrm{pair}-\mathrm{cr}(G) \leq \mathrm{cr}(G)$. And it's also obvious since we have the following trivial lemma.

LEMMA. For any graph G, there exists a drawing satisfying that there are $\mathrm{cr}(G)$ crossings, and every pair of edges crosses at most once. □

In the following sections, we discuss some details of the relationship of these four kinds of crossing numbers.

2 PLANARITY

It's obvious that a simple graph G is planar if and only if $\mathrm{cr}(G) = 0$. In this section, we will show that a simple graph G is planar if and only if any one of these four crossing numbers of G is zero. It's enough to prove the following two theorem.

THEOREM 2. For any simple graph G, if the crossing number of G is zero, then the rectilinear crossing number of G is zero, i.e. $\mathrm{cr}(G) = 0 \Rightarrow \mathrm{lin} - \mathrm{cr}(G) = 0$.

THEOREM 3. For any simple graph G, if the odd-crossing number of G is zero, then the crossing number of G is zero, i.e. $\mathrm{odd} - \mathrm{cr}(G) = 0 \Rightarrow \mathrm{cr}(G) = 0$.

2.1 RECTILINEAR CROSSING NUMBER

First, we prove another somewhat stronger theorem in order to show Theorem 2.

THEOREM 4.[2] G is a simple planar graph, and in a planar drawing D of G

1. Every interior vertex, i.e. not the vertices of the unbounded face, has degree at least 3.
2. Each bounded face of D is simply connected.
3. The intersection of two bounded faces is empty or connected.

Then there is a convex polygonal drawing D' of G, i.e. each bounded face in D' is a convex polygon.

The key point of the proof is the following lemma.

LEMMA. If drawing D of planar graph G, which has at least two bounded faces, satisfies 1), 2) and 3), then there are two bounded faces of D, say A and B, such that A and B touch on an arc and for any other bounded face C of D the set $(A \cup B) \cap C$ is connected or empty.

PROOF OF THE LEMMA. Assume, for contradiction, that for every two bounded faces A and B which touch on an arc, there is a third bounded face C so that $(A \cup B) \cap C$ is not connected. Then the boundary of $A \cup B \cup C$ consists of two simple closed cycle, P_1 and P_2, and P_1 lies inside P_2.
Assume that the bounded faces A, B, and C satisfy the conditions above and the interior of P_1 has the minimum area over all possible A, B, C.
There must exist further faces interior to P_1, and at least one of them, say B', touches A on an arc. Thus there is a face C' such that $(A \cup B') \cap C'$ is not connected. The boundary of $A \cup B' \cup C'$ consists of two simple closed cycle, P'_1 and P'_2, and P'_1 lies inside P'_2.
Since both B' and C' lie inside P_1, P'_1 also lies inside P_1. Then the area of the interior of P'_1 is smaller than that of P_1, contradiction! □

PROOF OF THEOREM 4. We show it by induction on m, the number of bounded faces of D.
Basis. For $m = 1$, it is trivial.
Induction Step. Assume the theorem is true for $m - 1$ faces planar graph. Let D be the drawing, satisfying 1), 2) and 3), of simple planar graph G with m bounded faces.
By the lemma, there are two faces A and B touching on an arc v_1v_2, and $(A \cup B) \cap C$ is connected or empty for all bounded face C of D.
Remove edge v_1v_2 and make A and B into a single bounded face A'. Then there is a convex polygonal drawing D'' of $G - \{v_1v_2\}$, in which face A' is a convex polygon. Add the line segment v_1v_2 in this polygon, then we get a convex polygonal drawing D' of G. □

PROOF OF THEOREM 2. For any simple graph G, if $\mathrm{cr}(G) = 0$, then G is planar. In planar drawing D of G, each bounded face of D is simply connected, so it satisfies 2). For each interior vertex with degree 1, remove this vertex and the edge adjacent to it; for each interior vertex with degree 2, remove this vertex and make the two edges adjacent to it to one edge. For any two bounded faces A and B with not connected intersection, you can add several edges and cut A and B to several parts to make sure this won't happen. After these operations, we get a planar drawing D' of another simple graph G', which satisfies 1), 2) and 3). So that, by Theorem 4, we have a convex polygonal drawing D'' of G'. In D'', we delete the edges we've added, add back the vertices of degree 2 as the midpoints of the corresponding line segments, and then add the vertices of degree 1 and the edge adjacent to it. Finally, we have a drawing of G, in which every edge is represented by a line segment. Thus $\mathrm{lin}-\mathrm{cr}(G) = 0$. □

2.2 ODD-CROSSING NUMBER

In [9], Tutte has given these two theorems.

THEOREM 5.(KURATOWSKI'S THEOREM) A graph G is planar if and only if no subgraph of G is a subdivision of a K_5 or a $K_{3,3}$.

THEOREM 6. In any planar representation of a subdivision G of K_5 or $K_{3,3}$, there are two edges, derived from non-adjacent edges of K_5 or $K_{3,3}$, which cross odd number of times.

By these two theorems, we can know that if $\mathrm{odd}-\mathrm{cr}(G) = 0$, G doesn't contain a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G is planar. Thus $\mathrm{cr}(G) = 0$.

3 DIFFERENCE

We already know that for any simple graph G, $\mathrm{odd}-\mathrm{cr}(G) \leq \mathrm{pair}-\mathrm{cr}(G) \leq \mathrm{cr}(G) \leq \mathrm{lin}-\mathrm{cr}(G)$, and if any one of these four is equal to zero then all are equal to zero. Are they the same for any simple graph?

The problem that whether $\mathrm{pair}-\mathrm{cr}(G) = \mathrm{cr}(G)$ turns out to be quite challenging, and remains open. But there are simple graphs with $\mathrm{cr}(G) \neq \mathrm{lin}-\mathrm{cr}(G)$ or $\mathrm{odd}-\mathrm{cr}(G) \neq \mathrm{pair}-\mathrm{cr}(G)$.

3.1 COMPLETE GRAPH

Guy's conjecture (cf. [3]) said that the crossing number for the complete graph K_n is

$$\mathrm{cr}(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor \tag{1}$$

Here is a construction which shows that (1) is an upper bound. (cf. [4])

CONSTRUCTION. Partition the n vertices into u and v vertices with $u+v=n$ and $v=u$ or $v=u+1$. Arrange the u vertices on a circle, and the other v vertices on another circle, outside the first one. Connect any two pair of the u vertices by line segments, and that of the v vertices by arcs outside the outer circle. Connect one of the u vertices to each of the v vertices clockwise, then pick the next one of the u vertices anti-clockwise and connect it to each of the v vertices clockwise but start with the anti-clockwise one of the former start point. Repeat until all u vertices are connected to all v vertices. Then the number of crossings between the two circles $\mathrm{cr}'(K_n)$ is

$$\mathrm{cr}'(K_n) = \begin{cases} \frac{1}{6}u^2(u-1)(u-2), & \text{if } v=u \\ \frac{1}{6}(u+1)u(u-1)^2, & \text{if } v=u+1 \end{cases}$$

Adding these to $\binom{u}{4}+\binom{v}{4}$ gives that

$$\mathrm{cr}(K_n) \le \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor = \begin{cases} \frac{1}{64}n(n-2)^2(n-4), & \text{if } n \text{ is even} \\ \frac{1}{64}(n-1)^2(n-3)^2, & \text{if } n \text{ is odd} \end{cases}$$

□

For $n \le 10$, (1) has been proved in [5]

n	2	3	4	5	6	7	8	9	10
$\mathrm{cr}(K_n)$	0	0	0	1	3	9	18	36	60

And also in [5], Guy has shown that $\text{lin}-\text{cr}(K_8) = 19 > 18 = \text{cr}(K_8)$. So that there exists simple graph G such that $\text{lin}-\text{cr}(G) \neq \text{cr}(G)$.

Moreover, in [6], it was said that $\text{lin}-\text{cr}(K_n) = \text{cr}(K_n)$ if and only if $n \leq 7$ or $n = 9$.

3.2 MAP CROSSING NUMBERS

In [10], M. J. Pelsmajer, M. Schaefer, and D. Štefankovič have shown that odd-crossing number and pairwise crossing number are not the same. They defined a new kind of crossing number: map crossing number. They gave counterexample to $\text{odd}-\text{cr}(M) = \text{pair}-\text{cr}(M)$ for maps on the annulus, then translated the map counterexample into a finite family of simple graphs for which $\text{odd}-\text{cr}(G) < \text{pair}-\text{cr}(G)$.

THEOREM 7.[10] There are simple graphs G which satisfy

$$\text{odd}-\text{cr}(G) \leq (\frac{\sqrt{3}}{2} + o(1))\text{pair}-\text{cr}(G).$$

4 RESTRAINT

We proved that odd-crossing number, crossing number, and rectilinear crossing number are not the same. But how different are they?

It was shown by Bienstock and Dean in [7] that there are graphs with crossing number 4 and arbitrarily large rectilinear crossing number. Is this also true for odd-crossing number and crossing number, or for pairwise crossing number and crossing number?

At first, we give another theorem without proof, then we use this theorem to prove two restraint of crossing number with odd-crossing number and pairwise crossing number.

THEOREM 8.[1] For a fixed drawing of a simple graph G, let $G_0 \subseteq G$ denote the subgraph formed by all even edges (an *even* edge is a edge which crosses every other edge an even number of times). Then G can be drawn in such a way that the edges belonging to G_0 are not involved in any crossing.

THEOREM 9.[1] The crossing number of any simple graph G satisfies that $\mathrm{cr}(G) \leq 2(\mathrm{odd}-\mathrm{cr}(G))^2 \leq 2(\mathrm{pair}-\mathrm{cr}(G))^2$.

PROOF. Let D be a drawing of G with $m = \mathrm{odd} - \mathrm{cr}(G)$ pairs of edges that cross an odd number of times. Let $E_0 \subseteq E$ denote the set of even edge. Then $|E-E_0| \leq 2m$. By Theorem 8, there exists a drawing of G,in which no edge of E_0 is involved in any crossing. Choose a drawing D' with this property and minimum number of crossings, then any two edges cross at most once. Thus $\mathrm{cr}(G) \leq \binom{|E-E_0|}{2} \leq \binom{2m}{2} \leq 2m^2$. □

THEOREM 10.[8] The crossing number of any simple graph G satisfies that $\mathrm{cr}(G) \leq O(\frac{k^2}{\log k})$, where $k = \mathrm{pair} - \mathrm{cr}(G)$.

SKETCH OF PROOF. G is a simple graph.

1 Let D_0 be a drawing of G with k pairs of edges that cross. Let $t = \dfrac{1}{2}\log k$.

 (1) E_0 is the set of edges with no crossing.

 (2) E_1 is the set of edges crossing at most t edges.

 (3) E_2 is the set of edges crossing more than t edges.

2 Drawing D_1 is:

 (1) E_0 as in D_0.

(2) Any edge of E_1 crosses at most t other edges and minimize the number of crossings.

(3) Edges in E_2 satisfy

i. No crossing between E_0 and E_2.

ii. minimize the number of crossings between E_1 and E_2.

iii. minimize the number of crossings among E_2.

3 *Claim.* In D_1

(1) Any edge of E_1 crosses at most 2^t times with edges of E_1.

(2) Any edge of E_2 crosses any other edge at most once.

4 Finally, We have $\text{cr}(G) \leq O(2^t|E_1| + |E_2|(|E_1| + |E_2|)) \leq O(2^t 2k + \frac{2k}{t}2k) \leq O(\frac{k^2}{\log k})$. □

By these two restraint of crossing number, we know that a simple graph with fixed odd-crossing number (or pairwise crossing number) cannot have arbitrarily large crossing number.

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