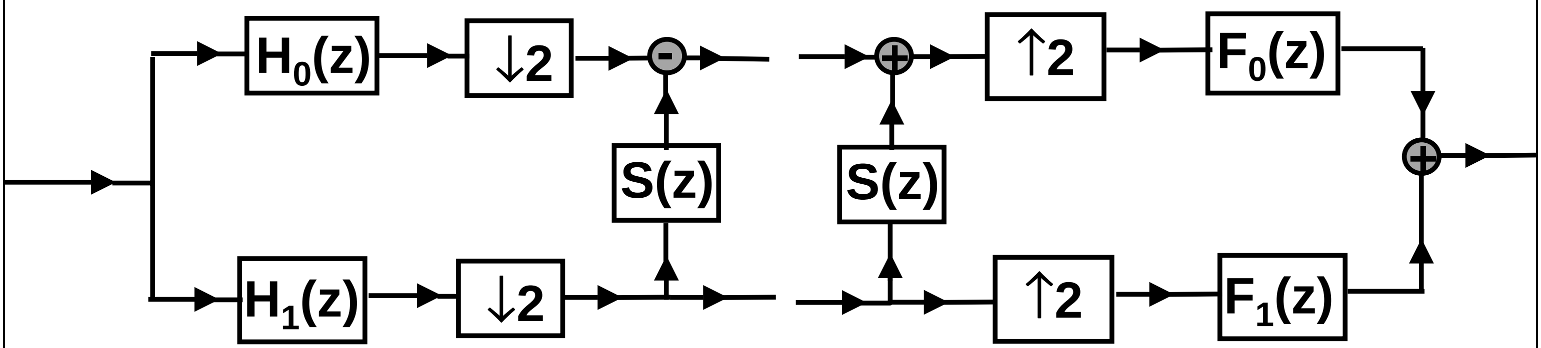
Course 18.327 and 1.130
Wavelets and Filter Banks

Lifting: ladder structure for filter banks; factorization of polyphase matrix into lifting steps; lifting form of refinement equation

Lifting

Basic idea:



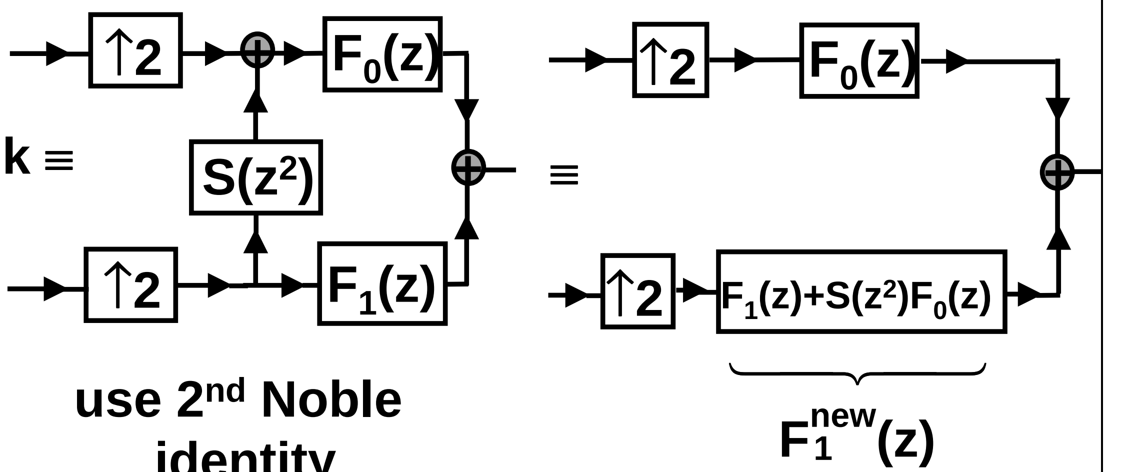
Filter bank is modified by a simple operation that preserves the perfect reconstruction property, regardless of the actual choice for S(z).

Advantages:

- **Leads to faster implementation of DWT**
- **Provides a framework for constructing wavelets on non-uniform grids.**

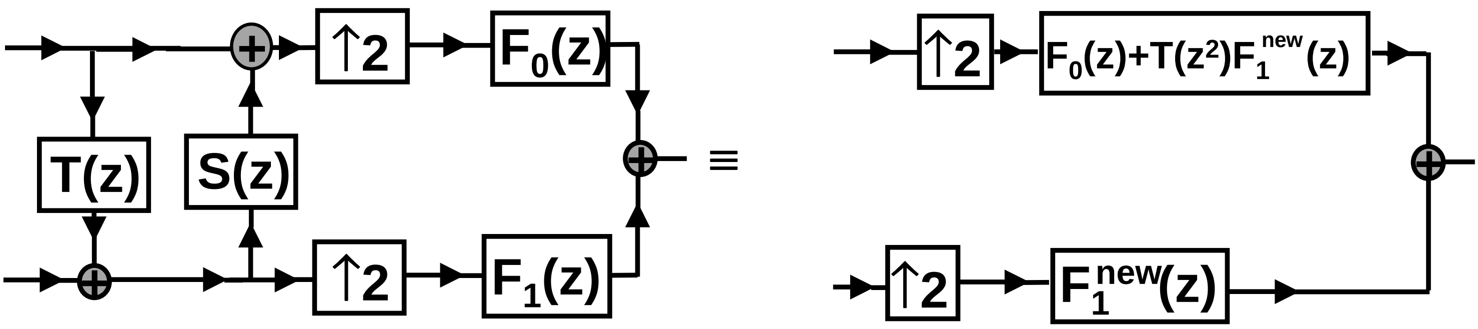
What are the effective filters in the modified filter bank?

Synthesis bank ≡



use 2nd Noble identity

So the effective highpass filter is $F_1(z) + S(z^2)F_0(z)$. The lowpass filter is unchanged. To modify the lowpass filter, add a second lifting step, e.g.



Consider

$$F_1^{new}(z) = F_1(z) + S(z^2)\,F_0(z)$$

i.e. $$f_1^{new}[n] = f_1[n] + \sum_k r[k]\, f_0[n-k]$$

where

$$r[k] = \begin{cases} s[k/2] & ;\ k \text{ even} \\ 0 & ;\ k \text{ odd} \end{cases}$$

$r[2k] = s[k]$

$r[2k+1] = 0$

So

$$f_1^{new}[n] = f_1[n] + \sum_k s[k]\, f_0[n-2k]$$

Then the corresponding wavelet is

$$w^{new}(t) = \sum_n f_1^{new}[n]\, \phi(2t-n)$$

$$= \sum_n f_1[n]\, \phi(2t-n) + \sum_k s[k] \sum_n f_0[n-2k]\, \phi(2t-n)$$

$$= w(t) + \sum_k s[k] \sum_\ell f_0[\ell]\phi(2t - 2k - \ell)$$

$$= w(t) + \sum_k s[k]\, \phi(t-k) \qquad \text{since } \phi(t) = \sum_\ell f_0[\ell]\phi(2t-\ell)$$

Lifting for wavelet bases

- **Lifting construction can be used to build a more complex set of scaling functions and wavelets from an initial biorthogonal set.**

e.g. lifting step S(z) gives

$$\phi^{new}(t) = \phi(t) \qquad (f_0[n] \text{ unchanged})$$

$$w^{new}(t) = w(t) - \sum_k s[k]\,\phi(t-k)$$

$$\tilde{\phi}^{new}(t) = \sum_n h_0[n]\,\tilde{\phi}^{new}(2t-n) + \sum_k s[k]\,\tilde{w}^{new}(t-k)$$

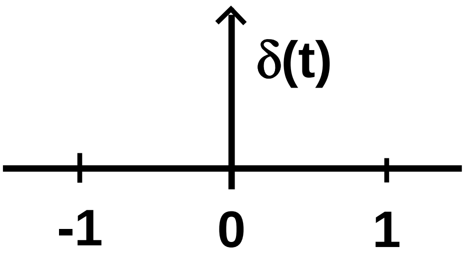
$$\tilde{w}^{new}(t) = \sum_n h_1[n]\tilde{\phi}^{new}(2t-n) \qquad (h_1[n] \text{ unchanged})$$

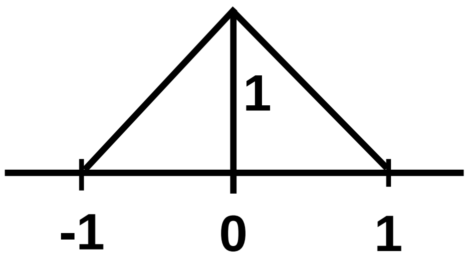
Example:

$H_0(z) = \sqrt{2}$ $\qquad$ $F_0(z) = \sqrt{2}\,\{\tfrac{1}{4}z + \tfrac{1}{2} + \tfrac{1}{4}z^{-1}\}$

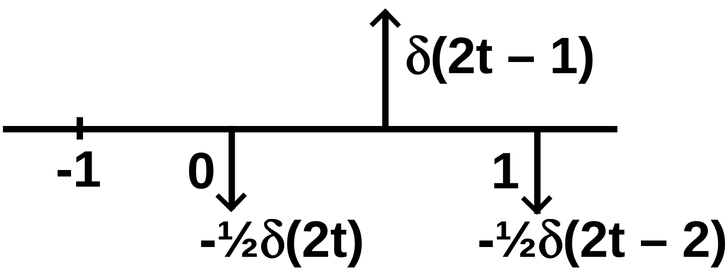
$H_1(z) = \sqrt{2}\{-\tfrac{1}{4} + \tfrac{1}{2}z - \tfrac{1}{4}z^{2}\}$ $\qquad$ $F_1(z) = \sqrt{2}\,z^{-1}$

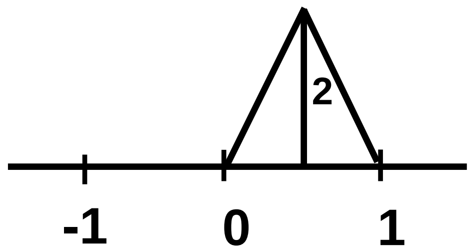
Scaling functions and wavelets are





$\tilde{\phi}(t) = 2\tilde{\phi}(2t)$ $\qquad$ $\phi(t) = \tfrac{1}{2}\phi(2t+1) + \phi(2t) + \tfrac{1}{2}\phi(2t-1)$





$\tilde{w}(t) = -\tfrac{1}{2}\tilde{\phi}(2t) + \tilde{\phi}(2t-1) - \tfrac{1}{2}\tilde{\phi}(2t-2)$ $\qquad$ $w(t) = 2\phi(2t-1)$

Biorthogonality/PR conditions are easy to verify, but what about zeros at π?

$F_0(z)$ **has double zero at** π

$H_0(z)$ **has no zeros at** $\pi \rightarrow$ **bad**

i.e. w(t) has no vanishing moments

Lifting step to add vanishing moments to the synthesis wavelet:

Suppose that the new wavelet has the form

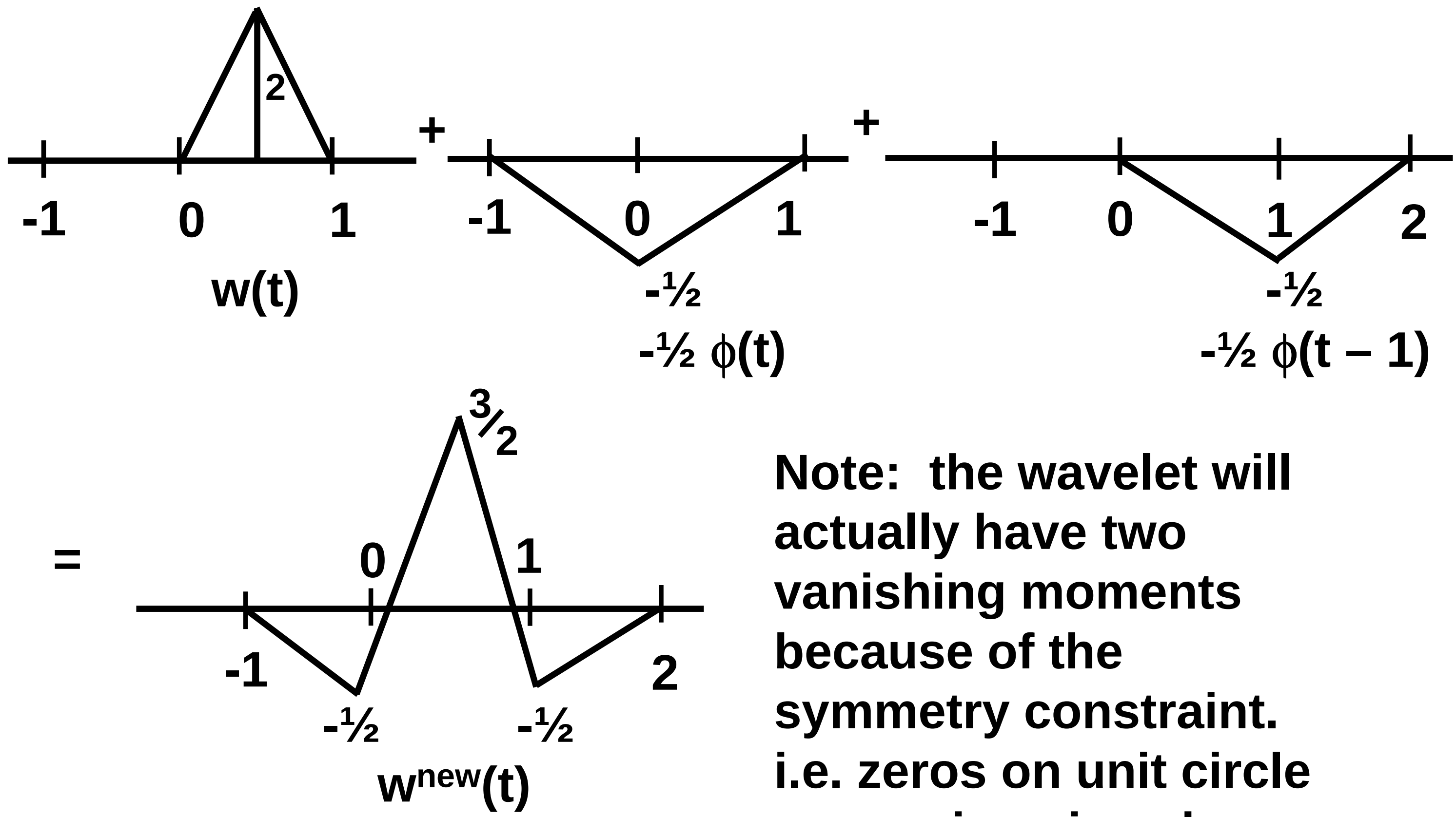
$$w^{new}(t) = w(t) - \alpha\,\phi(t) - \alpha\,\phi(t-1)$$

Goal is to make the zero^th moment vanish

$$\int_{-\infty}^{\infty} w^{new}(t)dt = \tfrac{1}{2} \bullet 1 \bullet 2 - \alpha \bullet 1 - \alpha \bullet 1$$

$$= 0 \text{ when } \alpha = \tfrac{1}{2}$$

So the new wavelet is



Note: the wavelet will actually have two vanishing moments because of the symmetry constraint. i.e. zeros on unit circle appear in pairs when filter is symmetric.

What is $F_1^{new}(z)$?

New wavelet equation is

$$w^{new}(t) = w(t) - \tfrac{1}{2}\phi(t) - \tfrac{1}{2}\phi(t-1)$$
$$= 2\phi(2t-1) - \tfrac{1}{2}\{\tfrac{1}{2}\phi(2t+1) + \phi(2t) + \tfrac{1}{2}\phi(2t-1)\}$$
$$- \tfrac{1}{2}\{\tfrac{1}{2}\phi(2t-1) + \phi(2t-2) + \tfrac{1}{2}\phi(2t-3)\}$$
$$= -\tfrac{1}{4}\phi(2t+1) - \tfrac{1}{2}\phi(2t) + \tfrac{3}{2}\phi(2t-1) - \tfrac{1}{2}\phi(2t-2) - \tfrac{1}{4}\phi(2t-3)$$

So

$$F_1^{new}(z) = \sqrt{2}\{-\tfrac{1}{8}z - \tfrac{1}{4} + \tfrac{3}{4}z^{-1} - \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3}\}$$

This can be rewritten as

$$F_1^{new}(z) = \sqrt{2}\{\underbrace{z^{-1}}_{F_1(z)} + \underbrace{\frac{-(1+z^{-2})}{2}}_{S(z^2)} \underbrace{(\tfrac{1}{4}z + \tfrac{1}{2} + \tfrac{1}{4}z^{-1})}_{F_0(z)}\}$$

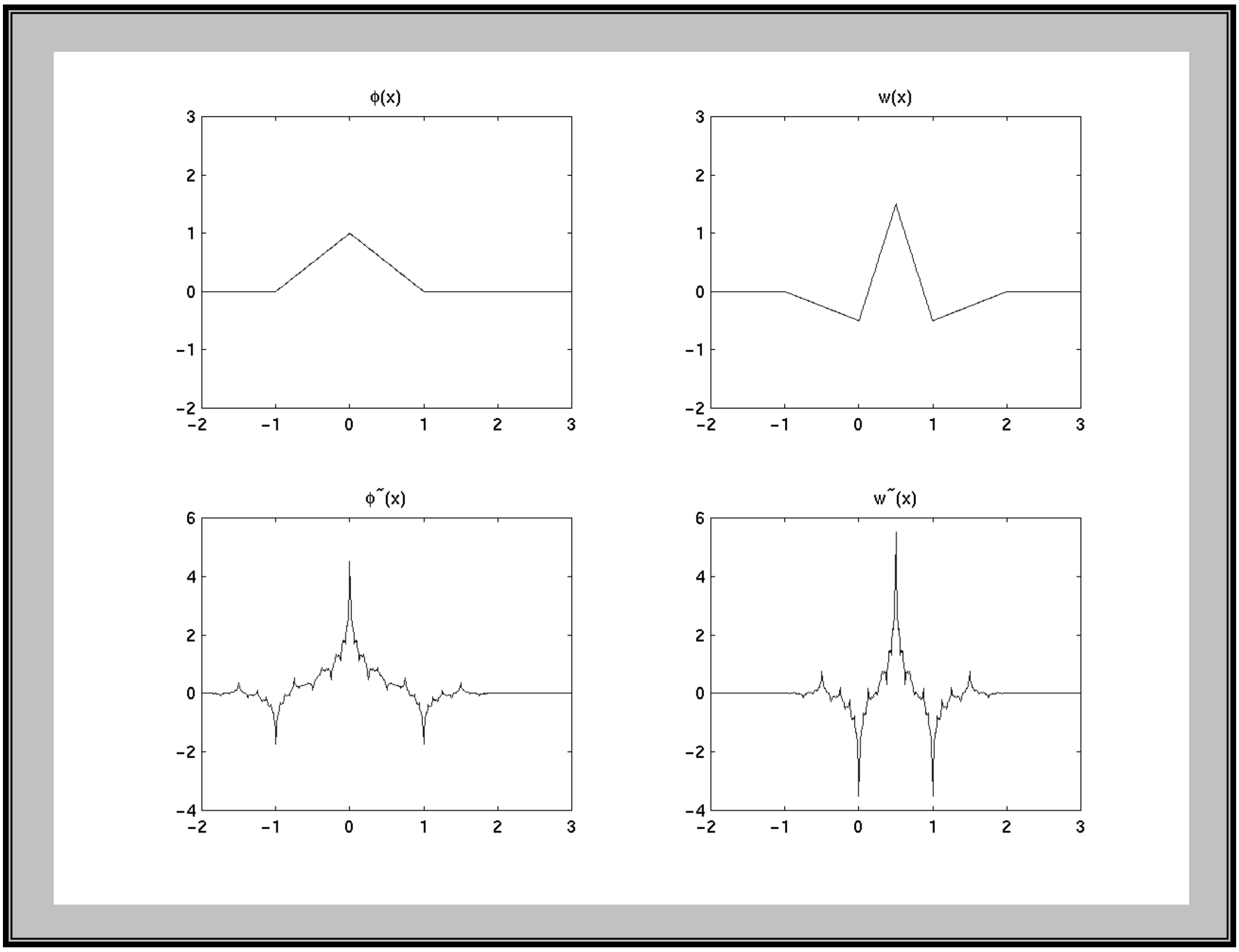
The new analysis lowpass filter is

$$H_0^{new}(z) = \sqrt{2}\left\{1 + \frac{(1 + z^{-2})}{2}\left(-\tfrac{1}{4} + \tfrac{1}{2}z - \tfrac{1}{4}z^{2}\right)\right\}$$

This can be written as

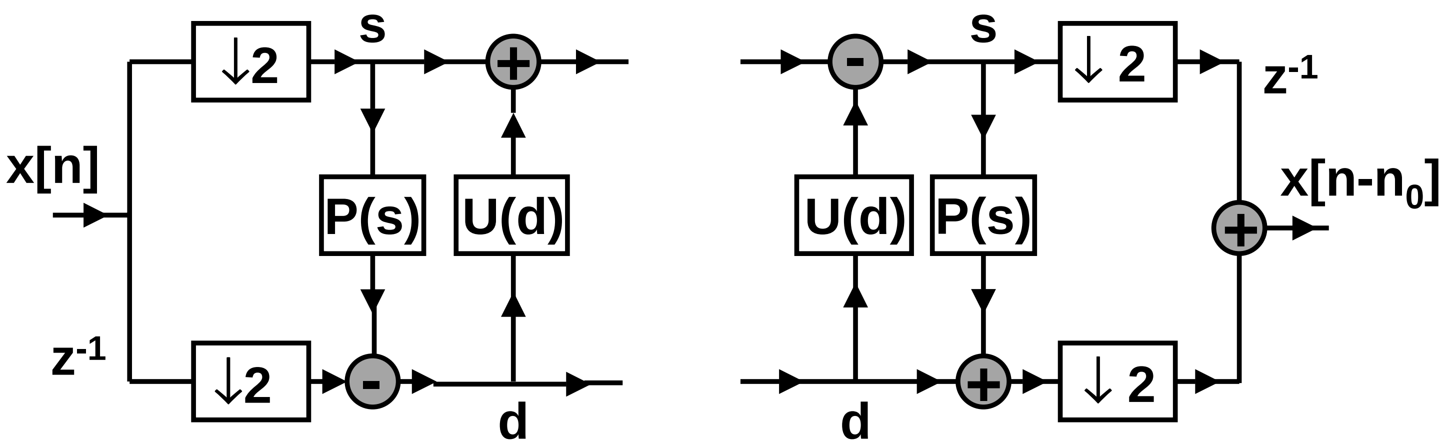
$$\left.\begin{aligned} H_0^{new}(z) &= \sqrt{2} \bullet \tfrac{1}{8}\,(1 + z)(1 + z^{-1})(-z + 4 - z^{-1}) \\ F_0(z) &= \sqrt{2} \bullet \tfrac{1}{4}\,(1 + z)(1 + z^{-1}) \end{aligned}\right\} \text{5/3 filter bank}$$

Symmetric 5/3 Wavelets



Efficient Implementations

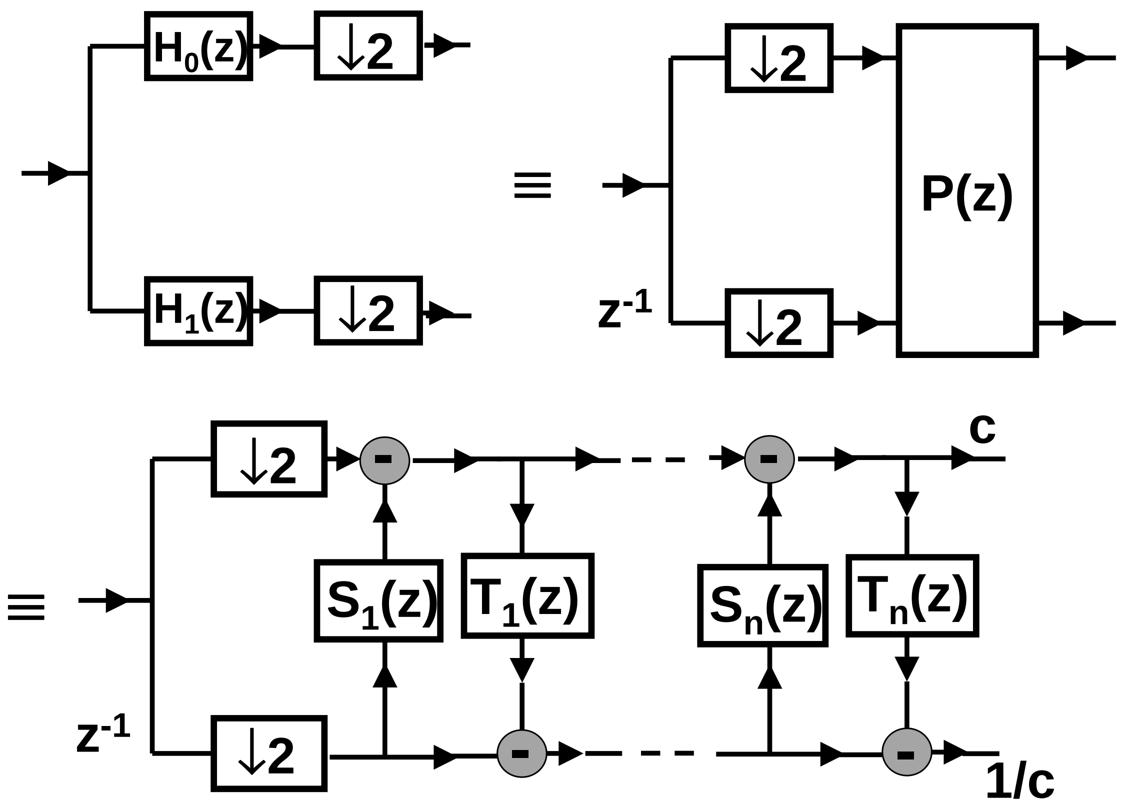
Ladder structure



P and U may be nonlinear e.g. truncation to integer

Factorization of Filter Bank into Lifting Steps (Daubechies & Sweldens)

Goal is to perform a change of representation of the form:



$$P(z) \equiv \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \prod_{i=n}^{1} \begin{bmatrix} 1 & 0 \\ -T_i(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -S_i(z) \\ 0 & 1 \end{bmatrix}$$

Approach: use Euclidean algorithm for greatest common divisor

1) Start with

$$A_0(z) = H_{0,even}(z)$$
$$B_0(z) = H_{0,odd}(z)$$

2) Then iterate

$$A_i(z) = B_{i-1}(z)$$

$$B_i(z) = A_{i-1}(z) \,\%\, B_{i-1}(z) = A_{i-1}(z) - Q_i(z)\, B_{i-1}(z)$$

remainder operator **quotient** $\frac{A_{i-1}(z)}{B_{i-1}(z)}$ **(non-unique)**

until i = n

$A_n(z) = c \quad \leftarrow \text{gcd}(H_{0,even}(z), H_{0,odd}(z))$

$B_n(z) = 0$

Matrix form of iteration:

$$\begin{bmatrix} A_i(z) & B_i(z) \end{bmatrix} = \begin{bmatrix} A_{i-1}(z) & B_{i-1}(z) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -Q_i(z) \end{bmatrix}$$

After n iterations:

$$\begin{bmatrix} c & 0 \end{bmatrix} = \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \end{bmatrix} \prod_{i=1}^{n} \begin{bmatrix} 0 & 1 \\ 1 & -Q_i(z) \end{bmatrix}$$

Invert this result to get

$$\begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} \prod_{i=n}^{1} \begin{bmatrix} Q_i(z) & 1 \\ 1 & 0 \end{bmatrix}$$

Suppose that n is even (n = 2m).

We can obtain a valid polyphase matrix of the form

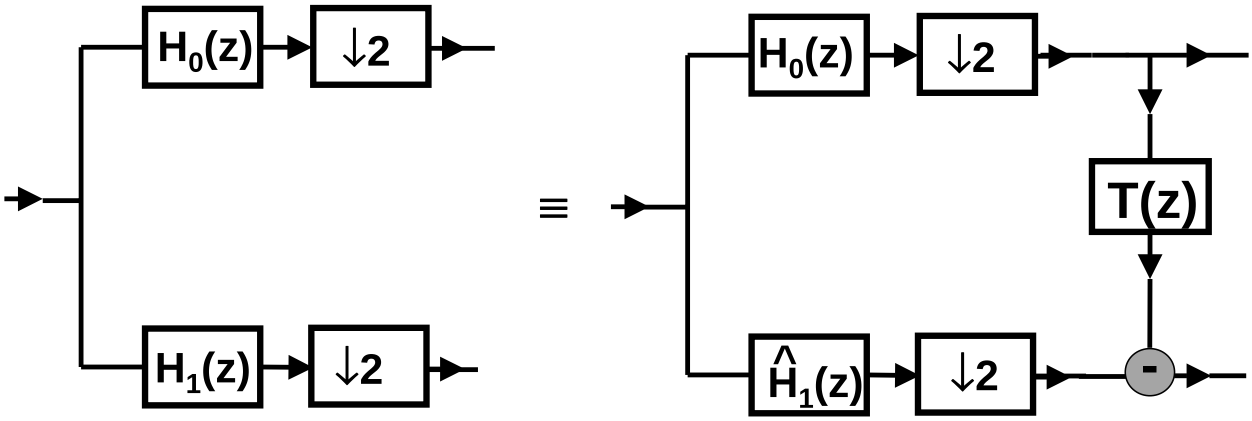
$$\hat{P}(z) = \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \prod_{i=2m}^{1} \begin{bmatrix} Q_i(z) & 1 \\ 1 & 0 \end{bmatrix}$$

Choice $1/c$ ensures that det $\hat{P}(z) = 1$

$$= \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ \hat{H}_{1,even}(z) & \hat{H}_{1,odd}(z) \end{bmatrix}$$

$\leftarrow$ $\hat{H}_1(z)$ gives P. R., but may not be the same as $H_1(z)$

To recover the original highpass filter, $H_1(z)$, from $\hat{H}_1(z)$, we introduce one more lifting step



$$H_1(z) = \hat{H}_1(z) - T(z^2)\, H_0(z)$$

So the polyphase matrix is

$$P(z) = \begin{bmatrix} 1 & 0 \\ -T(z) & 1 \end{bmatrix} \underbrace{\begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \prod_{i=2m}^{1} \begin{bmatrix} Q_i(z) & 1 \\ 1 & 0 \end{bmatrix}}_{\hat{P}(z)}$$

$$= \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -c^2T(z) & 1 \end{bmatrix} \prod_{k=m}^{1} \begin{bmatrix} Q_{2k}(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Q_{2k-1}(z) & 1 \\ 1 & 0 \end{bmatrix}$$

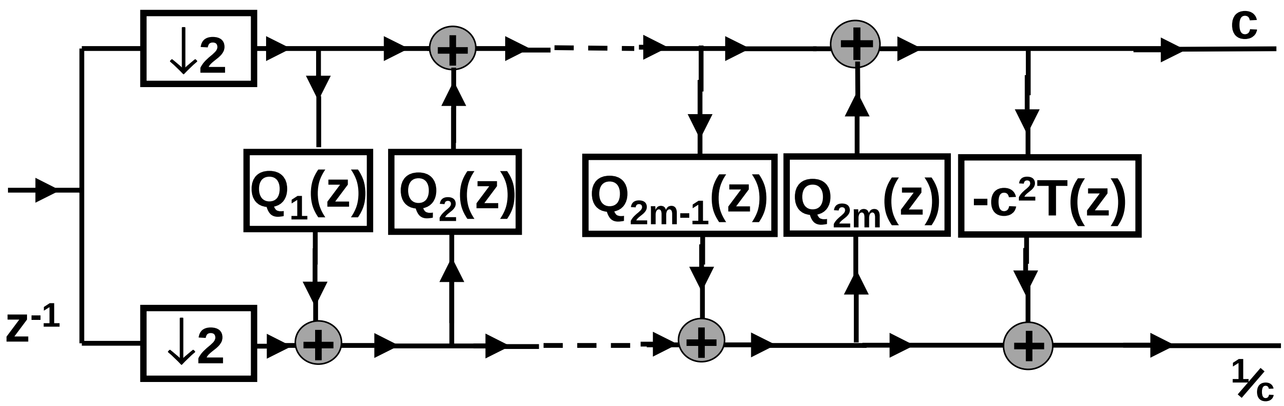
Rewrite each factor as a permutation of columns or rows

$$\begin{bmatrix} Q_{2k}(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Q_{2k-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & Q_{2k}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q_{2k-1}(z) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & Q_{2k}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q_{2k-1}(z) & 1 \end{bmatrix}$$

So

$$P(z) = \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -c^2T(z) & 1 \end{bmatrix} \prod_{k=m}^{1} \begin{bmatrix} 1 & Q_{2k}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q_{2k-1}(z) & 1 \end{bmatrix}$$



Example: Haar

$$H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}) \qquad H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$

$$A_0(z) = H_{0,even}(z) = \frac{1}{\sqrt{2}}$$

$$B_0(z) = H_{0,odd}(z) = \frac{1}{\sqrt{2}}$$

$$A_1(z) = B_0(z) = \frac{1}{\sqrt{2}} = c$$

$$B_1(z) = A_0(z) \% B_0(z) = 0$$

$$Q_1(z) = A_0(z) / B_0(z) = 1$$

$$\hat{P}(z) = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\hat{H}_1(z) = \frac{2}{\sqrt{2}}$$

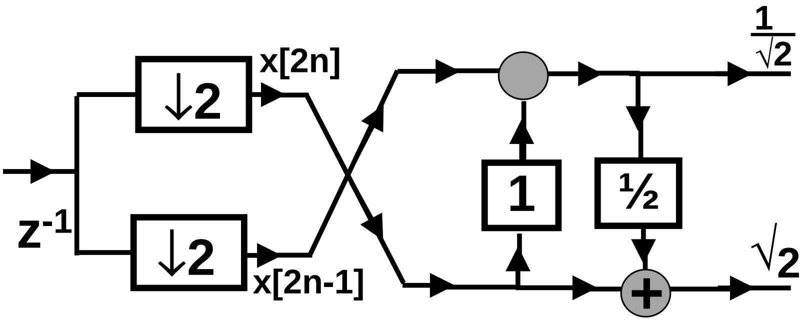
$$H_1(z) = \hat{H}_1(z) - T(z^2)\, H_0(z)$$

$$\frac{1}{\sqrt{2}}(1 - z^{-1}) = \frac{2}{\sqrt{2}} - T(z^2)\, \frac{1}{\sqrt{2}}(1 + z^{-1})$$

i.e. $T(z^2) = 1$

$$P(z) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -½ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -½ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

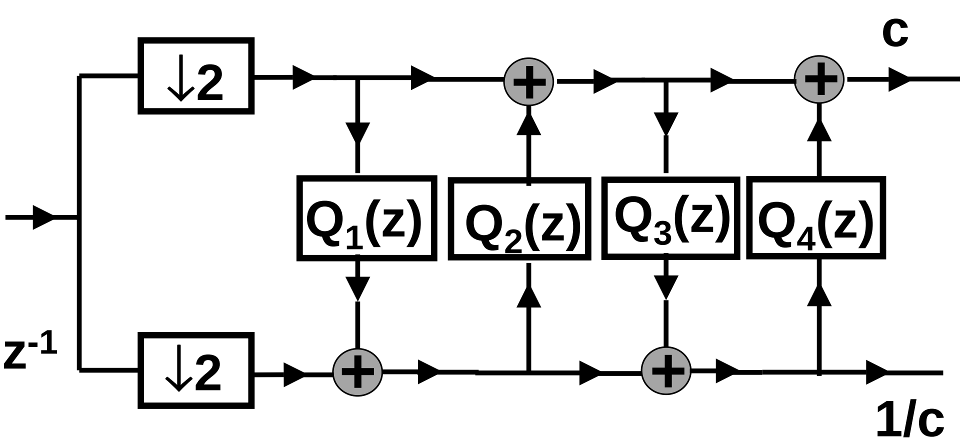


$$y_0[n] = \frac{1}{\sqrt{2}}(x[2n] + x[2n-1])$$

$$y_1[n] = \sqrt{2}(x[2n] - ½ \cdot \sqrt{2}\, y_0[n])$$

$$= \frac{1}{\sqrt{2}}(x[2n] - x[2n-1])$$

Factorization for 9/7 filter bank



$Q_1(z) = \alpha(1+z)$

$Q_2(z) = \beta(1+z^{-1})$

$Q_3(z) = \gamma(1+z)$

$Q_4(z) = \delta(1+z^{-1})$

$\alpha =$ -1.586134342

$\beta =$ -0.05298011854

$\gamma =$ 0.8829110762

$\delta =$ 0.4435068522

c = 1.149604398