Conservation Laws

$$u_t + (f(u))_x = 0 \quad \overset{\text{if } u \in C^1}{\Longleftrightarrow} \quad u_t + \underbrace{f'(u)}_{=c(u)} u_x = 0$$

Conservation form — Differential form

$\Updownarrow$

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)) \qquad f = \text{flux function}$$

Integral form

<u>Ex.</u>: Transport equation

$f(u) = cu \Rightarrow c(u) = f'(u) = c$

<u>Ex.</u>: Burgers' equation

$f(u) = \frac{1}{2}u^2 \Rightarrow c(u) = f'(u) = u$

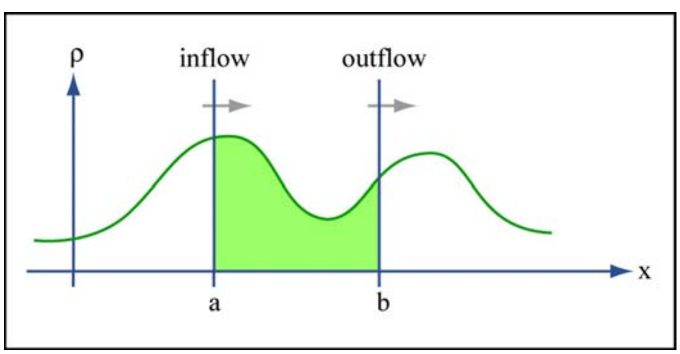
$u_t + uu_x = 0$

Model for fluid flow

Material derivative: $\frac{Du}{Dt} = u_t + (u \cdot \nabla)u \overset{1D}{=} u_t + uu_x$

<u>Ex.</u>: Traffic flow

$\rho(x,t) = \text{vehicle density} \left\{ \begin{array}{ll} \rho = 0 & \text{empty} \\ \rho = 1 & \text{packed} \end{array} \right\}$



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$m(t) = \int_a^b \rho(x,t)dx = \text{number of vehicles in } [a,b]$

$\frac{d}{dt}m(t) = f(\rho(a,t)) - f(\rho(b,t))$

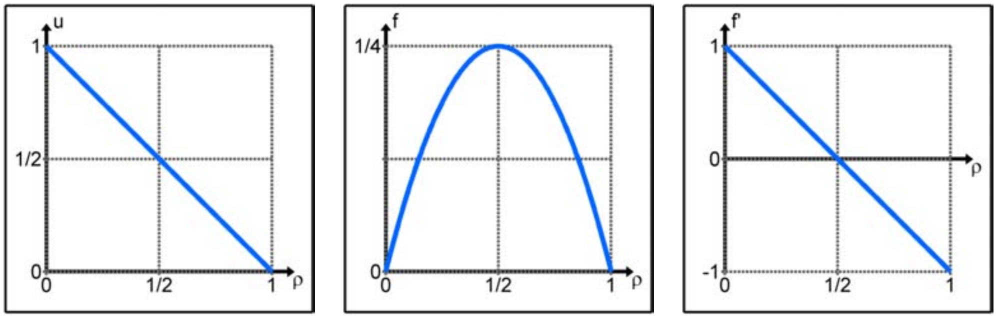
Influx — Outflux

Equation: $\rho_t + (f(\rho))_x = 0$ where $f(\rho) = \underbrace{v}_{\text{vehicle velocity}} \cdot \rho$

Velocity function $v = v(\rho) = 1 - \rho$

Flux function $\Rightarrow f(\rho) = \rho(1 - \rho)$

Velocity of information $c(\rho) = f'(\rho) = 1 - 2\rho$



Method of Characteristics

$$\begin{cases} u_t + f'(u)u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

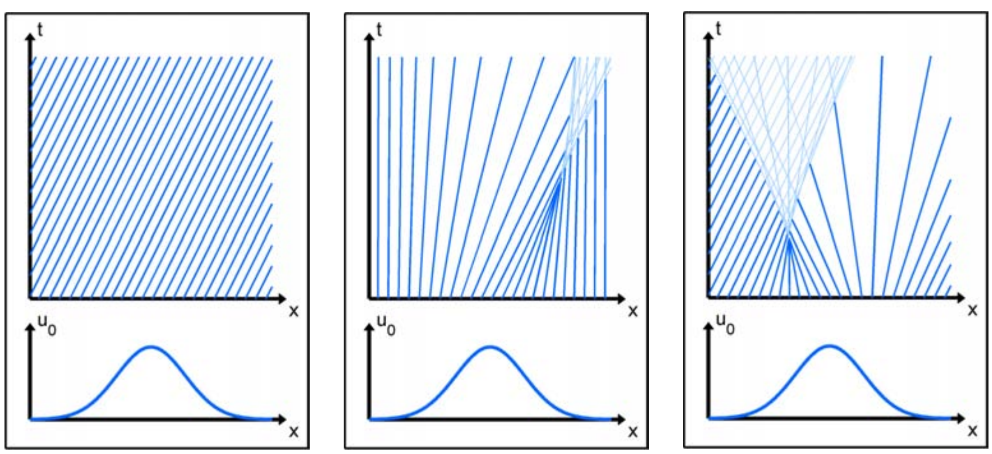
Follow solution along line $x_0 + ct$, where $c = f'(u_0(x_0))$.

$$\frac{d}{dt}u(x+ct,t) = cu_x(x+ct,t) + u_t(x+ct,t)$$
$$= \underbrace{(c - f'(u(x+ct,t)))}_{=0} \cdot u_x(x+ct,t) = 0$$

$\Rightarrow u(x+ct,t) = \text{ constant } = u(x_0,0) = u_0(x_0)$.

Ex.:

Transport Burgers' Traffic



Characteristic lines intersect $\Rightarrow$ shocks

Weak Solutions

$$(*) \begin{cases} u_t + (f(u))_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

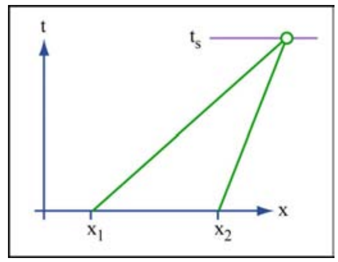
Solution for $u_0 \in C^1$ smooth until characteristics cross.

$$x_1 + f'(u_0(x_1)) \cdot t = x_2 + f'(u_0(x_2)) \cdot t$$

$$\Rightarrow t = -\frac{x_2 - x_1}{f'(u_0(x_2)) - f'(u_0(x_1))} = -\frac{1}{(f' \circ u_0)'(\tilde{x})} \quad \tilde{x} \in [x_1, x_2]$$

$$= -\frac{1}{f''(u_0(\tilde{x}))u_0'(\tilde{x})}$$

$$\Rightarrow t_s = -\frac{1}{\inf_x f''(u_0(x))u_0'(x)}$$



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Solution for $t > t_s$:

Weak formulation

$$(**) \int_0^\infty \int_{-\infty}^{+\infty} u\varphi_t + f(u)\varphi_x \, dx \, dt = -\int_{-\infty}^{+\infty} [u\varphi]_{t=0} \, dx \quad \forall \underbrace{\varphi \in C_0^1}_{\text{Test function, } C^1 \text{ with compact support}}$$

If $u \in C^1$ ("classical solution"), then $(*) \Leftrightarrow (**)$

Proof: integration by parts.

In addition, $(**)$ admits discontinuous solutions.

Riemann Problem

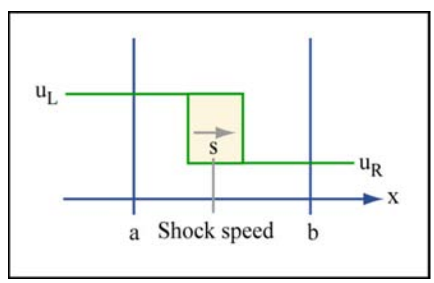
$$u_0(x) = \begin{Bmatrix} u_L & x < 0 \\ u_R & x \geq 0 \end{Bmatrix}$$

$$(u_L - u_R) \cdot s = \frac{d}{dt} \int_a^b u(x,t) \, dx$$

$$= f(u_L) - f(u_R)$$

$$\Rightarrow \boxed{s = \frac{f(u_R) - f(u_L)}{u_R - u_L}}$$

Rankine-Hugoniot Condition for shocks



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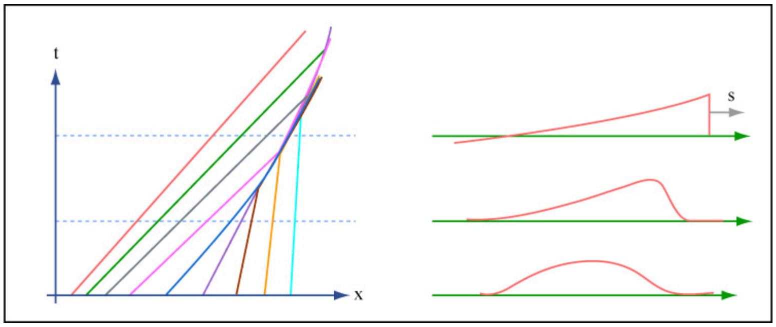
Ex.: Burgers'



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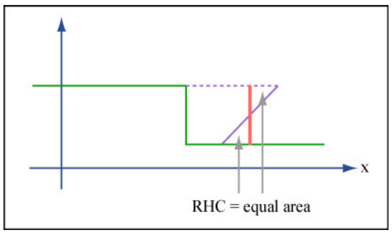
$$s = \frac{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2}{u_R - u_L} = \frac{u_L + u_R}{2}$$



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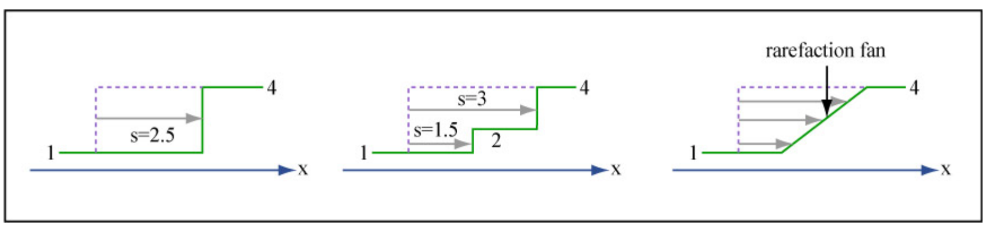
Replace breaking wave by shock

Rarefactions

Ex.: Burgers'

Many weak solutions...



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This is what physics yields (stable w.r.t. small perturbations)

Entropy Condition to single out unique weak solution:

- Characteristics go into shock:

  $f'(u_L) > s > f'(u_R)$

- Solution to $u_t + (f(u))_x = 0$ is limit of $u_t + (f(u))_x = \nu u_{xx}$ as $\nu \to 0$

  "Vanishing viscosity"

- Many more...

All equivalent.