16 Singular perturbations

The *singular perturbation* is the bogeyman of applied mathematics. The fundamental problem is to ask: when can you neglect a term in a continuous equation? The answer is not always obvious and, amongst other things, this was the reason why early attempts to understand the theory of flight failed so dramatically. Before progressing towards this, we shall begin with a few examples of singular perturbations.

16.1 Magnetization

A magnet is composed of atoms, each of which has a molecular spin. The energetics of the interaction between the spins is that each spin produces a magnetic field which tries to align the neighboring spins. A popular microscopic model for a magnet imagines the spins confined to a regular lattice, and then ascribes an energy

$$U = \sum_{i \sim j} J_{ij} \boldsymbol{s}_i \cdot \boldsymbol{s}_j \tag{414}$$

where $i \sim j$ indicates a summation over nearest neighbors. A typical approximation is to take the sum over only the nearest neighbors of a given spin and to take the interaction constants J_{ij} to be a constant.

If one assumes that the local spins vary on a length scale much longer than a lattice spacing, then it is possible to derive a *macroscopic* analogue of the above energy. A complete derivation of this includes the effect of random thermal fluctuations and is beyond the scope of this course. For simplicity we consider just a one dimensional array of atoms for which the energy is

$$U[M(x)] = \int \left[\nu \left(\frac{dM}{dx} \right)^2 + f(M) \right] dx, \tag{415a}$$

where M is the magnitude of the local magnetization, which depends on the average spin in a small region, and

$$f(M) = -bM^2 + cM^4. \tag{415b}$$

Physically, ν punishes gradients in magnetization. If $b < 0$ then we have a *paramagnet*, with $M = 0$ being the minimum energy configuration. Otherwise if $b > 0$ then we have a *ferromagnet*, with minima at $\pm\sqrt{b/(2c)}$.

Using the calculus of variations the function $M(x)$ that minimises the energy satisfies

$$\nu \frac{d^2M}{dx^2} - bM + 2cM^3 = 0. \tag{416}$$

We already know that

$$M = 0, \pm\sqrt{\frac{b}{2c}} \tag{417}$$

are three constant solutions. Now if $\nu = 0$ there is no penalty for orientation change throughout the system, and for $b > 0$ the entire system has magnetization $\pm\sqrt{\frac{b}{2c}}$ with any orientation. If the system has boundaries then the magnetization must match the boundary conditions, but is otherwise free to be orientated however it wants.

What happens, however, if $\nu \neq 0$? If we multiply both sides of (3) by M', then the equilibrium condition can be integrated to give

$$\frac{dM}{dx} = \frac{1}{\nu}\sqrt{bM^2 - cM^4 + k}, \tag{418}$$

where k is a constant. Rearranging this one obtains

$$\int \frac{\nu\, dM}{\sqrt{bM^2 - cM^4 + k}} = \int dx. \tag{419}$$

Solving this, subject to the appropriate boundary conditions, one finds that *domain boundaries* arise. These are transition regions in which the magnetization flips from the value imposed at one boundary to that at the other boundary. In the limit of $\nu \to 0$, these domain boundaries become infinitely sharp.

So now you start to get an idea of the problem. If $\nu = 0$ then the orientation of the spins throughout the system is arbitrary, except at the boundaries, which are fixed. However, even for extremely small non-zero ν (e.g., 10^{-100}), we have completely different behaviour and obtain extended regions of uniform magnetization separated by a sharp domain boundary. The different behaviour arises because if ν is nonzero the entire system is forced to match the imposed boundary conditions at the edges. Setting $\nu = 0$ is therefore called a singular perturbation.

16.2 An elementary algebraic equation

As another example of a singular perturbation, consider the solution of the algebraic equation

$$bx + c = 0. \tag{420}$$

The solution is simply $x = -c/b$. Now we make a small change, and consider the equation

$$\epsilon x^2 + bx + c = 0, \tag{421}$$

Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4\epsilon c}}{2\epsilon}. \tag{422}$$

In the limit $\epsilon \to 0$

$$x \approx -\frac{c}{b}, -\frac{2b - 2\epsilon c}{2\epsilon}. \tag{423}$$

and the latter solution can be further approximated as $-b/\epsilon$ if ϵ is very small. If this term has some physical significance then you are in trouble. You cannot simply neglect the term ϵx^2 in the original problem.

16.3 An elementary differential equation

Let's consider the differential equation[23]

$$\epsilon \frac{d^2u}{dx^2} + \frac{du}{dx} = 1. \tag{424}$$

If ϵ is very small we might argue that we can neglect this term, the solution therefore being

$$u = x + C. \tag{425}$$

Alternatively, if we consider the full problem the solution is

$$u = A + x + Be^{-x/\epsilon}. \tag{426}$$

Imposing the boundary conditions $u(0) = 0, u(1) = 2$, for the full problem we determine A and B, and find that

$$u = x + \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \tag{427}$$

is the exact solution. We cannot apply both these boundary conditions to our approximate solution (as it is a first order equation), so we choose the 'outer' condition $u(1) = 2$. The approximate solution satisfying the outer condition is therefore

$$u = x + 1. \tag{428}$$

In the outer region the approximate solution and the true solution are very close. However, in a region close to $x = 0$ they differ greatly. We call this the *boundary layer*. It arises because the small parameter ϵ multiplies the highest derivative in the equation, and by ignoring this term we lower the order of the system and are unable to satisfy both boundary conditions.

We need to find an approximate 'inner' solution that matches the boundary condition at $x = 0$. To do so, we change the independent variable to

$$X = \frac{x}{\epsilon}. \tag{429}$$

This enables us to zoom in on the boundary layer. With this scaling the original equation becomes

$$\frac{1}{\epsilon}\frac{d^2u}{dX^2} + \frac{1}{\epsilon}\frac{du}{dX} = 1, \tag{430}$$

[23] See Acheson, pp. 269-271

so that to a first approximation the 'inner' solution satisfies

$$\frac{d^2u}{dX^2} + \frac{du}{dX} = 0. \tag{431}$$

Imposing the boundary condition at $X = 0$ gives

$$u = A(1 - e^{-X}) = A(1 - e^{-\frac{x}{\epsilon}}). \tag{432}$$

Finally, we require that as $X \to \infty$ the inner solution matches the outer solution in the limit $x \to 0$, so that $A = 1$.

We have thus been able to approximate the full solution in two parts, an inner and outer solution. Although we could solve for the full solution analytically, often this is not possible and we must resort to approximations like those used here. The inner solution is valid within a boundary layer of thickness ϵ and matches to the outer solution. Once again we see that ignoring the term multiplied by ϵ in the original problem is a *singular perturbation*; no matter how small ϵ is, there exists a region in which it has a significant affect on the solution. This idea was due to Prandtl, who first discovered it within the context of airplane flight. We will now take a bit of a digression to justify the concept of a boundary layer in fluid dynamics.

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