

4 Kepler's problem and Hamiltonian dynamics

Why do we study applied mathematics? Aside from the intellectual challenge, it is reasonable to argue that we do so to obtain an understanding of physical phenomena, and to be able to make predictions about them. Possibly the greatest example of this, and the origin of much of the mathematics we do, came from Newton's desire to understand the motion of the planets, which were known to obey Kepler's laws.

4.1 Kepler's laws of planetary motion

In the early seventeenth century (1609-1619) Kepler proposed three laws of planetary motion

- (i) The orbits of the planets are ellipses, with the Sun's centre of mass at one focus of the ellipse.
- (ii) The line joining a planet and the Sun describes equal areas in equal intervals of time.
- (iii) The squares of the periods of the planets are proportional to the cubes of their semi-major axes.

These laws were based on detailed observations made by Tycho Brahe, and put to rest any notion that planets move in perfectly circular orbits. However, it wasn't until Newton proposed his law of gravitation in 1687 that the origins of this motion were understood. Newton proposed that

“Every object in the Universe attracts every other object with a force directed along a line of centres for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation of the two objects.”

Based on this one statement, it is possible to derive Kepler's laws.

4.1.1 Second law

Kepler's second law is the simplest to derive, and is a statement that the angular momentum of a particle moving under a central force, such as gravity, is constant. By definition, the angular momentum \mathbf{L} of a particle with mass m and velocity \mathbf{u} is

$$\mathbf{L} = \mathbf{r} \wedge m \frac{d\mathbf{r}}{dt}, \quad (53)$$

where \mathbf{r} is the vector position of the particle. The rate of change of angular momentum is given by

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \wedge \mathbf{f} = \mathbf{r} \wedge f(r)\hat{\mathbf{r}} = 0, \quad (54)$$

where $f(r)\hat{\mathbf{r}}$ is the central force, depending only on the distance $r = |\mathbf{r}|$ and pointing in the direction $\hat{\mathbf{r}} = \mathbf{r}/r$. It can therefore be seen that the angular momentum of a particle moving under a central force is constant, a consequence of this being that motion takes place in a plane. The area swept out by the line joining a planet and the sun is half the area of the parallelogram formed by \mathbf{r} and $d\mathbf{r}$. Thus

$$dA = \frac{1}{2}|\mathbf{r} \wedge d\mathbf{r}| = \frac{1}{2}\left|\mathbf{r} \wedge \frac{d\mathbf{r}}{dt}dt\right| = \frac{L}{2m}dt, \quad (55)$$

where $L = |\mathbf{L}|$ is a constant. The area swept out is therefore also constant.

4.1.2 First law

To prove Keplers first law consider the sun as being stationary (i.e., infinitely heavy), and the planets in orbit around it. The equation of motion for a planet is

$$m\frac{d^2\mathbf{r}}{dt^2} = f(r)\hat{\mathbf{r}}. \quad (56)$$

In plane polar coordinates

$$\frac{d\mathbf{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \quad (57a)$$

$$\frac{d^2\mathbf{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \quad (57b)$$

In component form, equation (56) therefore becomes

$$m(\ddot{r} - r\dot{\theta}^2) = f(r), \quad (58a)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \quad (58b)$$

Putting (57a) into (53) gives

$$L = \left|\mathbf{r} \wedge m\frac{d\mathbf{r}}{dt}\right| = |mr^2\dot{\theta}|. \quad (59)$$

Thus

$$r^2\dot{\theta} = l, \quad (60)$$

where $l = L/m$ is the angular momentum per unit mass. Given a radial force $f(r)$, equations (58a) and (58b) can now be solved to obtain r and θ as functions of t . A more practical result is to solve for $r(\theta)$, however, and this requires the definition of a new variable

$$r = \frac{1}{u}. \quad (61)$$

Rewriting the equations of motion in terms of the new variable requires the identities

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -l\frac{du}{d\theta}, \quad (62a)$$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}. \quad (62b)$$

Equation (58a) becomes

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f\left(\frac{1}{u}\right). \quad (63)$$

This is the differential equation governing the motion of a particle under a central force. Conversely, if one is given the polar equation of the orbit $r = r(\theta)$, the force function can be derived by differentiating and putting the result into the differential equation. According to Newton's law of gravitation $f(r) = -k/r^2$, so that

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{ml^2}. \quad (64)$$

This has the general solution

$$u = A\cos(\theta - \theta_0) + \frac{k}{ml^2}, \quad (65)$$

where A and θ_0 are constants of integration that encode information about the initial conditions. Choosing $\theta_0=0$ and replacing u by the original radial coordinate $r = 1/u$,

$$r = \left(A\cos\theta + \frac{k}{ml^2} \right)^{-1} \quad (66)$$

which is the equation of a conic section with the origin at the focus. This can be rewritten in standard form

$$r = r_0 \frac{1 + \epsilon}{1 + \epsilon\cos\theta}, \quad (67)$$

where

$$\epsilon = \frac{Aml^2}{k}, \quad r_0 = \frac{ml^2}{k(1 + \epsilon)}. \quad (68)$$

ϵ is called the eccentricity of the orbit:

- $\epsilon = 0$ is a circle,
- $\epsilon < 1$ is an ellipse,
- $\epsilon = 1$ is a parabola and
- $\epsilon > 1$ is a hyperbola.

For an elliptical orbit, r_0 is the distance of closest approach to the sun, and is called the *perihelion*. Similarly,

$$r_1 = r_0(1 + \epsilon)/(1 - \epsilon) \quad (69)$$

is the furthest distance from the sun and is called the *aphelion*. Orbital eccentricities are small for planets, whereas comets have parabolic or hyperbolic orbits. Interestingly though, Halley's comet has a very eccentric orbit but, according to the definition just given, is not a comet!

4.1.3 Third law

To prove Kepler's third law go back to equation (55). Integrating this area law over time gives

$$A(\tau) = \int_0^\tau dA = \frac{l\tau}{2}, \quad (70)$$

where τ is the period of the orbit. The area A of an ellipse can also be written as $A = \pi ab$ where a and b are the semi-major and semi-minor axes, yielding

$$\frac{l\tau}{2} = \pi ab \quad (71)$$

The ratio of a and b can be expressed in terms of the eccentricity

$$\frac{b}{a} = \sqrt{1 - \epsilon^2}. \quad (72)$$

Using this expression to substitute b in (71)

$$\tau = \frac{2\pi a^2}{l} \sqrt{1 - \epsilon^2} \quad (73)$$

The length of the major axis is

$$2a = r_0 + r_1 = \frac{2ml^2}{k(1 - \epsilon^2)}. \quad (74)$$

Squaring (71) and replacing gives

$$\tau^2 = \frac{4\pi^2 m}{k} a^3, \quad (75)$$

confirming Kepler's 3rd law.

4.2 Hamiltonian dynamics of many-body systems

The Kepler problem is essentially a two-body problem. In the remainder of this course, we will be interested in classical (non-quantum) systems that consist of $N \gg 2$ particles. The complete microscopic dynamics of such systems is encoded in their Hamiltonian

$$H = \sum_{n=1}^N \frac{\mathbf{p}_n^2}{2m_n} + U(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad (76a)$$

where m_n , $\mathbf{p}_n(t)$ and $\mathbf{x}_n(t)$ denote the mass, momentum and position of the n th particle. The first contribution on the rhs. of Eq. (76a) is the kinetic energy, and U is the potential energy. For our purposes, it is sufficient to assume that we can decompose (76a) into a sum of pair interactions

$$U(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{2} \sum_{n,k:n \neq k} \Phi(\mathbf{x}_n, \mathbf{x}_k). \quad (76b)$$

Given H , Newton's equations can be compactly rewritten as

$$\dot{\mathbf{x}}_n = \nabla_{\mathbf{p}_n} H, \quad \dot{\mathbf{p}}_n = -\nabla_{\mathbf{x}_n} H. \quad (77)$$

That this so-called Hamiltonian dynamics is indeed equivalent to Newton's laws of motion can be seen by direct insertion, which yields

$$\dot{\mathbf{x}}_n = \frac{\mathbf{p}_n}{m_n}, \quad \dot{\mathbf{p}}_n = m_n \ddot{\mathbf{x}}_n = -\nabla_{\mathbf{x}_n} U. \quad (78)$$

An important observation is that many physical systems obey certain conservation laws. For instance, the Hamiltonian (76a) itself remains conserved under the time-evolution (77)

$$\begin{aligned} \frac{d}{dt} H &= \sum_n [(\nabla_{\mathbf{p}_n} H) \cdot \dot{\mathbf{p}}_n + (\nabla_{\mathbf{x}_n} H) \cdot \dot{\mathbf{x}}_n] \\ &= \sum_n [(\nabla_{\mathbf{p}_n} H) \cdot (-\nabla_{\mathbf{x}_n} H) + (\nabla_{\mathbf{x}_n} H) \cdot \nabla_{\mathbf{p}_n} H] \equiv 0. \end{aligned} \quad (79)$$

which is just the statement of energy conservation. Other important examples of conserved quantities are total linear momentum and angular momentum,

$$\mathbf{P} = \sum_n \mathbf{p}_n, \quad \mathbf{L} = \sum_n \mathbf{x}_n \wedge \mathbf{p}_n \quad (80)$$

if the pair potentials Φ only depend on the distance between particles.

There exists a deep mathematical connection between such invariants and symmetries of the underlying Hamiltonian, known as Noether's theorem. For example, energy conservation is a consequence of the fact that the Hamiltonian (76a) is not explicitly time-dependent and, hence, invariant under time translations. Similarly, conservation of linear momentum is linked to spatial translation invariance and conservation of angular momentum to rotational invariance.

For the remainder of this course, it will be important to keep in mind that microscopic symmetries and conservation laws must be preserved in coarse-grained macroscopic continuum descriptions.

4.3 Practical limitations

Deriving Kepler's laws required us to solve a second-order linear ordinary differential equation, which was obtained by considering the idealised case in which a single planet is orbiting the sun. If we consider a more realistic problem in which several planets orbit the sun, all interacting with each other via gravity, the problem becomes analytically intractable. Indeed, for just two planets orbiting the sun one encounters the celebrated 'three-body problem', for which there is no general analytical solution. Lagrange showed that there are some solutions to this problem if we restrict the planets to move in the same plane, and assume that the mass of one of them is so small as to be negligible. In the absence of an explicit solution to the 'three body problem' one must use ideas from 18.03 to calculate fixed points of the equations and investigate their stability. However, now you see the problem. The critical number of equations for complicated things to happen is three (for ODE's), and yet any

relevant problem in the world contains many more than three degrees of freedom. Indeed a physical problem typically contains 10^{23} interacting particles (Avogadro's number), which is so great a number that it is unclear if the mathematical techniques described above are of any use. The central aim of this course is to make theoretical progress towards understanding systems with many degrees of freedom. To do so we shall invoke the "continuum hypothesis", imagining that the discrete variable (e.g. the velocity of a particular molecule of fluid) can be replaced with a continuum (e.g. the velocity field $\mathbf{v}(\mathbf{x}, t)$). There are many subtleties that arise in trying to implement this idea, among them;

- (i) How does one write down macroscopic descriptions in terms of microscopic constants in a systematic way? It would be terrible to have to solve 10^{23} coupled differential equations!
- (ii) Forces and effects that *a priori* appear to be small are not always negligible. This turns out to be of fundamental importance, but was not recognised universally until the 1920's.
- (iii) The mathematics of how to solve 'macroscopic equations', which are nonlinear partial differential equations, is non-trivial. We will need to introduce many new ideas.

In tackling these problems we will spend a lot of time doing fluid mechanics, the reason being that it is by far the most developed field for the study of these questions. Experiments are readily available and the equations of motion are very well known (and not really debated!). Furthermore, fluid dynamics is an important subject in its own right, being relevant to many different scientific disciplines (e.g. aerospace engineering, meteorology, coffee cups). We will also introduce other examples (e.g. elasticity) to show the generality of the ideas.

4.4 Suggestions

For more details on Kepler's laws, and a java applet to let you play with them, go to

<http://csep10.phys.utk.edu/astr161/lect/history/kepler.html>

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