Lecture 12: Black-Scholes-Merton and Beyond

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Dynamic Hedging

In the previous lecture we considered how to hedge risk for a single time period τ given the probability distribution of price, the payoff function $y(x)$, a current price x_0 and an interest rate r. All we need to do is to find the best linear fit

$$\widetilde{w}_0^* = \widetilde{u}_0^* + \phi^* \widetilde{x}_0, \tag{1}$$

for the payoff function $y(x) = w(x, t+\tau)$ (here $\widetilde{x}_0 = x_0 e^{r\tau}$, $\widetilde{w}_0 = w_0 e^{r\tau}$, and t is the current moment of time).

From now on, we assume that r is risk-free (which actually is not the case). Now we are going to repeat this process for N periods of time. In particular, we are interested in those cases when we can eliminate any risk and determine $w(x,t)$ uniquely. Basically, there are only two suitable cases.

The first one is the so-called *Binomial Model* (or *Binomial Tree*). Here at each step we have only two possible outcomes of x. Thus, linear fit (1) can be found in a unique way because there is only one line passing trough two points.

The second one is the Black-Scholes-Merton case. It is the continuum limit with the time step τ tending to zero, and, thus, $\delta x \to 0$. Obviously, here our best linear fit degenerates into a tangent to the payoff function which is unique (provided we have a nice payoff function). There are some concerns when we let $\tau \to 0$. For instance, in this case transaction cost grows to infinity. Another shortcoming is that the underlying asset takes independent steps after some finite correlation time; thus, we cannot assume that our process is Markovian.

Binomial Tree

Let x_+ and x_- denote two possible outcomes for the current price x. Similarly, let w_+ and w_- be the corresponding payoffs. Then the equation for the fitting line is:

$$\widetilde{w}_0 = \frac{(\widetilde{x}_0 - x_-)w_+ + (x_+ - \widetilde{x}_0)w_-}{x_+ - x_-}, \tag{2}$$

which implies that the hedge ratio is:

$$\phi(t) = \frac{w_+ - w_-}{x_+ - x_-}.$$

(Here again $\widetilde{x}_0 = x_0 e^{r\tau}$ and $\widetilde{w}_0 = w_0 e^{r\tau}$.)

In the following sections we will solve the equation (2) backwards in time.

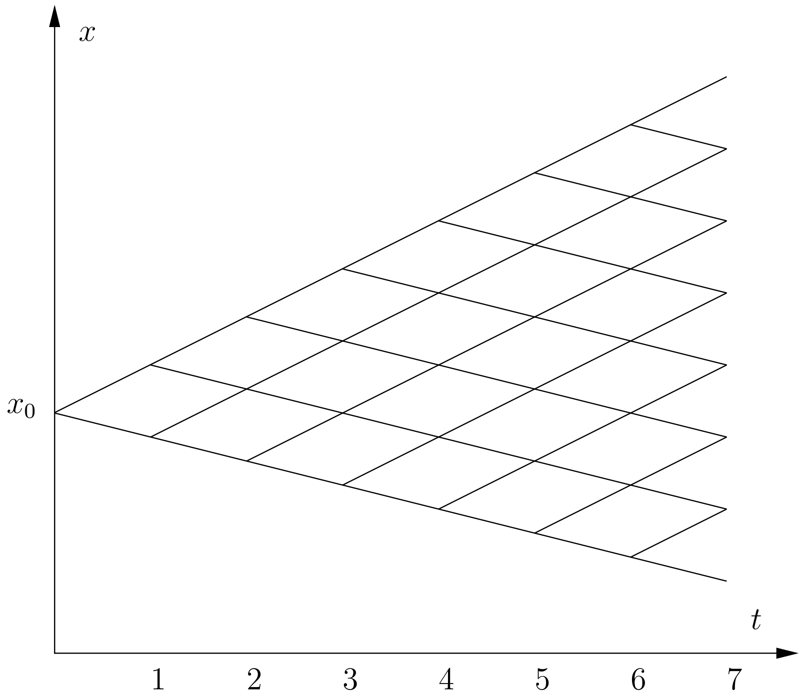

Figure 1: Binomial Tree

Continuum Limit

Normal (Additive) Stochastic Process for x_t

Let us consider the process where $x_\pm = x_0 \pm a$. Assuming that $r = 0$, we can rewrite the equation (2) as follows:

$$w(x,t) = \frac{1}{2}\left(w(x+a,t+\tau) + w(x-a,t+\tau)\right). \tag{3}$$

It is important to note that although the probabilities of $x_\pm$ could be different, we would still have the same equation (3).

Now we can consider the process as a Bernoulli random walk with probabilities 1/2 in backward time. Transforming (3), we get:

$$w(x,t) - w(x,t+\tau) = \frac{1}{2}\left(w(x+a,t+\tau) - 2w(x,t+\tau) + w(x-a,t+\tau)\right)$$

$$\begin{aligned}\left(w(x,t+\tau) - \tau\frac{\partial w}{\partial t} + \dots\right) - w(x,t+\tau) \quad = \quad & \frac{1}{2}\left(w(x,t+\tau) + a\frac{\partial w}{\partial x} + \frac{a}{2}\frac{\partial^2 w}{\partial x^2} + \dots\right. \\ & \left. -2w(x,t+\tau) + w(x,t+\tau) - a\frac{\partial w}{\partial x} + \frac{a}{2}\frac{\partial^2 w}{\partial x^2} + \dots\right).\end{aligned}$$

Here all partial derivatives are evaluated at the point $(x, t+\tau)$.

Now, omitting all higher-order terms (assuming that $\tau \to 0$ and $a \to 0$), we obtain the diffusion equation:

$$-\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}, \tag{4}$$

where

$$D = \frac{a^2}{2\tau}$$

is a diffusion coefficient.

Thus, we see that the option price 'diffuses' in backward time from the known payoff at $t = T$.

Lognormal (Multiplicative) Process for x_t

Here we will consider the binomial model with $x_\pm = xe^{\mu\pm\sigma\sqrt{\tau}}$. Expanding these outcomes we have:

$$\begin{aligned} x_\pm &= x\left(1 + \mu\tau \pm \sigma\sqrt{\tau} + \frac{\sigma^2\tau}{2} + \ldots\right) \\ &= x\left(1 + \bar{\mu}\tau \pm \sigma\sqrt{\tau} + \ldots\right), \end{aligned}$$

where $\bar{\mu} = \mu + \frac{\sigma^2}{2}$ is a noise induced drift.

Then the relative change of x is:

$$\frac{\Delta x}{x} = \Delta\left(\log x\right) = \bar{\mu}\tau \pm \sigma\sqrt{\tau}.$$

The quantities $\widetilde{w}_0$ and $\widetilde{x}_0$ can be represented as follows:

$$\begin{aligned} \widetilde{w}_0 &= e^{r\tau}w(x,t) = (1 + r\tau)w(x,t), \\ \widetilde{x}_0 &= e^{r\tau}x_0 = (1 + r\tau)x_0. \end{aligned}$$

Now we apply the binomial model for each time step τ:

$$(x_+ - x_-)\widetilde{w}_0 = (\widetilde{x}_0 - x_-)w_+ + (x_+ - \widetilde{x}_0)w_-$$

Substituting $\widetilde{w}_0$, $\widetilde{x}_0$ and $x_\pm$:

$$\left(2\sigma\sqrt{\tau}x\right)(1 + r\tau)w(x,t) = (r - \bar{\mu})\tau(w_+ - w_-) + \sigma\sqrt{\tau}x(w_+ + w_-),$$

where

$$w_\pm = w(x_\pm, t + \tau) = w(x, t + \tau) + (\bar{\mu}\tau \pm \sigma\sqrt{\tau})x\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + \frac{1}{2}\sigma^2\tau x^2\left.\frac{\partial^2 w}{\partial x^2}\right|_{(x,t+\tau)} + \ldots.$$

Thus,

$$w_+ - w_- = 2\sigma\sqrt{\tau}x\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + \ldots$$

and

$$w_+ + w_- = 2w(x, t+\tau) + 2\bar{\mu}\tau x\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + \sigma^2\tau x^2\left.\frac{\partial^2 w}{\partial x^2}\right|_{(x,t+\tau)} + \ldots.$$

This gives us:

$$(1 + r\tau)w(x,t) = (r - \bar{\mu})\tau x\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + w(x, t+\tau) + \bar{\mu}\tau x\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + \frac{\sigma^2}{2}\tau x^2\left.\frac{\partial^2 w}{\partial x^2}\right|_{(x,t+\tau)},$$

and the Black-Scholes miracle occurs: the drift (or expected return in x_t) drops out. However, $\bar{\mu}$ is present in higher-order terms.

Now, simplifying the obtained equation, we get:

$$\frac{w(x,t) - w(x,t+\tau)}{\tau} + rw(x,t) = rx\left.\frac{\partial w}{\partial x}\right|_{(x,t+\tau)} + \frac{\sigma^2 x^2}{2}\left.\frac{\partial^2 w}{\partial x^2}\right|_{(x,t+\tau)}.$$

Taking the limit with $\tau \to 0$, we obtain the *Black-Scholes equation*:

$$\frac{\partial w}{\partial t} + rx\frac{\partial w}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2 w}{\partial x^2} = rw. \tag{5}$$

Recall that we are assuming that $w(x,t)$ is *independent* of measure of risk expected return μ and r is a *risk-free* rate.

Risk-Neutral Valuation

Let us eliminate dimensions from the Black-Scholes equation. Namely, let us introduce the following variables:

$$\begin{aligned}
\bar{t} &= \frac{T-t}{T}, \\
\bar{x} &= \log\frac{x}{k}, \\
\bar{w} &= e^{r(T-t)}w, \\
\bar{r} &= rT, \\
\bar{\sigma}^2 &= \sigma^2 T.
\end{aligned}$$

Then, the partial derivatives have to be:

$$\begin{aligned}
\frac{\partial}{\partial \bar{x}} &= x\frac{\partial}{\partial x}, \\
\frac{\partial}{\partial \bar{t}} &= -\frac{1}{T}\frac{\partial}{\partial t}.
\end{aligned}$$

Now, equation (5) becomes:

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \left(\bar{r} - \frac{\bar{\sigma}^2}{2}\right)\frac{\partial \bar{w}}{\partial \bar{x}} + \frac{\bar{\sigma}^2}{2}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}. \tag{6}$$

The Green function (solution for the initial condition $\bar{w}(\bar{x},0) = \delta(\bar{x})$) for (6) is:

$$\bar{G}(\bar{x},\bar{t}) = \frac{1}{\sqrt{2\pi\bar{\sigma}^2\bar{t}}}\exp\left[-\frac{\left(\bar{x} + \bar{t}(\bar{r} - \bar{\sigma}^2/2)\right)^2}{2\bar{\sigma}^2\bar{t}}\right], \tag{7}$$

which is a normal distribution with mean $\bar{t}(\bar{r} - \bar{\sigma}^2/2)$ and variance $\bar{\sigma}^2\bar{t}$. Having this, we can write a solution with the arbitrary initial condition $\bar{y}(\bar{x})$ as a convolution:

$$\bar{w}(\bar{x},\bar{t}) = \left(\bar{G} \star \bar{y}\right)(\bar{x},\bar{t}),$$

which can be simplified as follows:

$$\bar{w}(\bar{x},\bar{t}) = \langle \bar{y}(\bar{x},\bar{t}) \rangle_{\bar{G}} .$$

Putting the dimensions back, we have the Green function for the Black-Scholes equation (5)

$$G(x,t) = e^{r(t-T)} L(x,t),$$

where $L(x,t)$ is a lognormal density with expected return r and volatility σ^2 which solves the following SDE for a lognormal process from t to T:

$$dx = r\,x\,dt + \sigma\,x\,dz$$

(Recall that in lecture 10 we showed that the mean of such a lognormal random variable with expected rate of return m and volatility σ has expected value $x_o e^{(m+\sigma^2/2)(T-t)}$, and here $m = r - \sigma^2/2$.) In terms of the Green function, the general solution can be written as an expectation of the payoff with respect to the lognormal process

$$w(x,t) = e^{r(t-T)} \langle y(x) \rangle.$$

This appears to be the same as Bachelier's fair price, equal to the expected payoff (discounted at the risk-free interest rate), except for one subtle difference: The mean rate of return μ in the lognormal process for the underlying asset has been replaced by r, the risk-free rate! This is again the "Black-Scholes mirable" caused by the hedging procedure, neglected by Bachelier, which removes any dependence on the mean return relative to the risk-free rate. Rather than solving the Black-Scholes PDE, therefore, we can instead applying the simple Bachelier "fair-game" principle, replacing μ with r. This procedure of *risk neutral valuation* is a powerful and widely used tool in options pricing and hedging.

In reality, however, a perfect hedge is not possible, and risk neutral valuation is only a first approximation. The presence of residual risk can be treated by various methods such as the Bouchaud-Sornette theory introduced in the previous lecture. In the continuum limit, this leads to perturbations of the Black-Scholes equation and corrections to risk-neutal valuation.

For more details, see the solutions to Problem Set 3 and the final project of Ken Gosier from 2001, *Derivatives Pricing and Hedging with Residual Risk*, both available online.