Problem Set Number 8, 18.385j/2.036j
MIT (Fall 2014)

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November 5, 2014

Due Fri., November 14, 2014.

1 Problems 08.07.05/06/07 - Strogatz (Another driven overdamped system)

Statement for problems 08.07.05/06/07

A. By considering an appropriate Poincaré map, prove that the system

$$\frac{d\theta}{dt} + \sin\theta = \sin t \tag{1.1}$$

has at least two periodic solutions. *Can you say anything about their stability?*

B. Give a mechanical interpretation for equation (1.1).

C. Plot a computer generated phase portrait for the system in (1.2). Check that the answer agrees with the results in part **A.**

Hint. Regard the system as a vector field on a cylinder:

$$\frac{dt}{dt} = 1 \qquad \text{and} \qquad \frac{d\theta}{dt} = \sin t - \sin\theta. \tag{1.2}$$

Sketch the nullclines and thereby infer the shape of certain key trajectories that can be used to bound the periodic solutions. For instance, sketch the trajectory through $\boldsymbol{P = (t,\, \theta) = \frac{1}{2}\,\pi\,(1,\, 1)}$.

2 Problem 141003 - Newton's Method in the complex plane

Statement for problem 141003

Suppose that you want to solve an equation, $g(x) = 0$. Then you can use *Newton's method*, which is as follows: Assume that you have a "reasonable" guess, x_0, for the value of a root. Then the sequence $\boldsymbol{x_{n+1} = f(x_n)}$, $\boldsymbol{n \geq 0}$, where

$$\boldsymbol{f(x) = x - \frac{g(x)}{g'(x)}}, \tag{2.1}$$

converges (very fast) to the root.

Remark 2.0.1 (The idea). *Assume an approximate solution $g(x_a) \approx 0$. Write $x_b = x_a + \delta x$ to improve it, where δx is small. Then $0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a)\,\delta x \;\Rightarrow\; \delta\, x \approx -\frac{g(x_a)}{g'(x_a)}$, and (2.1) follows.*
Of course, if x_0 is not close to a root, the method may not converge. Even if it converges, it may converge to a root that is far away from x_0, not necessarily the closest root. In this problem we investigate the behavior of Newton's method in the complex plane, *for arbitrary starting points.*

Consider iterations of the map in the complex plane generated by Newton's method for the roots of $z^3 - 1 = 0$. That is

$$z_{n+1} = f(z_n) = \left(\frac{2}{3} + \frac{1}{3\,z_n^3}\right) z_n, \quad n \geq 0, \tag{2.2}$$

where $0 < |z_0| < \infty$ is arbitrary. Note that

$$\boldsymbol{\zeta_1 = 1, \quad \zeta_2 = e^{i\,2\,\pi/3} = \frac{1}{2}\,(-1 + i\,\sqrt{3}),} \quad \text{and} \quad \boldsymbol{\zeta_3 = e^{i\,4\,\pi/3} = \frac{1}{2}\,(-1 - i\,\sqrt{3}),} \tag{2.3}$$

are the roots of $z^3 = 1$.

Your tasks: Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every [1] initial point z_0, draw a colored dot at the position of z_0, where the colors are picked as follows:

$$z_n \to \zeta_1, \text{ cyan.} \qquad z_n \to \zeta_2, \text{ magenta.} \qquad z_n \to \zeta_3, \text{ yellow.} \qquad \text{No convergence, black.} \tag{2.4}$$

What do you see? Do blow ups of the limit regions between zones.

Hint. Deciding that the sequence converges is easy: once z_n gets "close enough" to one of the roots, then the very design of Newton's method guarantees convergence. Thus, given a z_0, compute z_N for some large N, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some "small" tolerance δ — which does not have to be very small, in fact $\delta = 0.25$ is good enough. You can get pretty good pictures with $N = 50$ iterations on a 150×150 grid. A larger N is needed when refining near the boundary between zones.

Hint. If you use MatLab, do not plot "points". Instead, plot "regions", where the color of each pixel is decided by z_0 — use the command image$(x,\, y,\, C)$ *to plot. Why? Because using points leaves a lot of unpainted space in the figure, and gives much larger file sizes.*

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3 Coupled oscillators #01

Statement: Coupled oscillators #01

In this problem we *present an example of the process described in* § 4.1, and consider the *coupling of two oscillators with a stable, and strongly attracting, limit cycle each.* The oscillators are very simple, with trivial

[1] Numerically this means: choose a sufficiently fine grid in a rectangle, and pick every point in the grid. For example, select the square $-2 < x < 2$ and $-2 < y < 2$, where $z_0 = x + i\,y$.

equations in polar coordinates. This simplifies the analysis enormously, but the principles illustrated here are valid for the coupling of more generic oscillators.

Consider the following equations for two coupled oscillators

$$\dot{x_j} = -\omega_j\, y_j + \lambda_j\,(R_j^2 - x_j^2 - y_j^2)\, x_j + F_j(x_1,\, y_1,\, x_2,\, y_2), \tag{3.1}$$
$$\dot{y_j} = \omega_j\, x_j + \lambda_j\,(R_j^2 - x_j^2 - y_j^2)\, y_j + G_j(x_1,\, y_1,\, x_2,\, y_2), \tag{3.2}$$

where $j = 1$ or $j = 2$, and
(a) $\boldsymbol{\omega_j > 0}$, $\boldsymbol{R_j > 0}$ and $\boldsymbol{\lambda_j > 0}$, are constants, with $\boldsymbol{\lambda_j \gg 1}$,
(b) F_j and G_j are some functions — these are the coupling terms.

Using the fact that $\boldsymbol{\lambda_j \gg 1}$, **write reduced equations for the two phases $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$,** defined by $x_j = r_j\, \cos\theta_j$ and $y_j = r_j\, \sin\theta_j$, where $r_j = \sqrt{x_j^2 + y_j^2}$. In particular, consider the following cases

1. **What form do the reduced equations take when** the F_j and G_j are only functions of the variables $\boldsymbol{\eta = x_1\, x_2 + y_1\, y_2}$ and $\boldsymbol{\xi = y_1\, x_2 - x_1\, y_2}$.
2. **What form do the reduced equations take when** $\boldsymbol{G_1 = G_2 = 0}$, $\boldsymbol{F_1 = -\alpha\, \frac{R_1}{R_2}\, x_2}$, and $\boldsymbol{F_2 = -\beta\, \frac{R_2}{R_1}\, x_1}$ — where α and β are constants.

Hint. Write the equations in polar coordinates.[2] *Then consider what happens in a neighborhood of the limit cycles for the two oscillators when de-coupled — i.e.: r_j not too far from R_j. In this context, argue*[3] *that the dependence on the radial variables can be made trivial.*

4 Notes: coupled oscillators, phase locking, etc.

These are notes with facts useful for the problems. They are not a problem.

4.1 On phases and frequencies

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by $\boldsymbol{\vec{x}_1 = \vec{F}_1(\omega_1\, t)}$ and $\boldsymbol{\vec{x}_2 = \vec{F}_2(\omega_2\, t)}$, where $\boldsymbol{\vec{x}_1}$ and $\boldsymbol{\vec{x}_2}$ are the vectors of variables for each of the two systems, the $\boldsymbol{\vec{F}_j}$ *are periodic functions of period* $2\,\pi$, and the ω_j are constants (related to the limit cycle periods by $\omega_j = 2\,\pi/T_j$). In the un-coupled system, the two limit cycle orbits make up a *stable attracting invariant torus* for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. *Then the stable attracting invariant torus survives for the coupled system.*[4] The solutions (on this torus) can be (approximately) represented by

$$\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2), \tag{4.1}$$

[2] Recall that $\boldsymbol{r_j\, \dot{r}_j = x_j\, \dot{x}_j + y_j\, \dot{y}_j}$ and $\boldsymbol{r_j^2\, \dot{\theta}_j = x_j\, \dot{y}_j - y_j\, \dot{x}_j}$.
[3] Use arguments similar to the one introduced to describe relaxation oscillations, e.g.: for the van der Pol equation. Another example occurs when justifying that inertial terms can be neglected in the limit of a large viscosity.
[4] With a (slightly) changed shape and position.

where $\boldsymbol{\theta_1 = \theta_1(t)}$ and $\boldsymbol{\theta_2 = \theta_2(t)}$ satisfy some equations, of the general form

$$\dot{\theta}_1 = \omega_1 + K_1(\theta_1,\, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1,\, \theta_2). \tag{4.2}$$

Here K_1 and K_2 are the "projections" of the coupling terms along the oscillator limit cycles. For example, take $K_1(\theta_1,\, \theta_2) = \sin\theta_1\, \cos\theta_2$ and $K_2(\theta_1,\, \theta_2) = \sin\theta_2\, \cos\theta_1$. Another example is the one in § 8.6 of Strogatz' book (*Nonlinear Dynamics and Chaos*), where a model system with

$$K_1(\theta_1,\, \theta_2) = -\kappa_1\, \sin(\theta_1 - \theta_2) \text{ and } K_2(\theta_1,\, \theta_2) = \kappa_2\, \sin(\theta_1 - \theta_2)$$

is introduced, with constants $\kappa_1,\, \kappa_2 > 0$. Note that:

1. In (4.2), K_1 and K_2 must be $2\,\pi$-periodic functions of θ_1 and θ_2.
2. The **phase space for (4.2) is the invariant torus $\mathcal{T}$**, on which θ_1 and θ_2 are the angles. We can also think of $\mathcal{T}$ as a $2\,\pi \times 2\,\pi$ square with its opposite sides identified. On $\mathcal{T}$ a **solution is periodic** if and only if $\theta_1(t+T) = \theta_1(t) + 2\,n\,\pi$ and $\theta_2(t+T) = \theta_2(t) + 2\,m\,\pi$, where $T > 0$ is the period, and both n and m are integers.
3. In the "Coupled oscillators # 01" problem an example of the process leading to (4.2) is presented.
4. The $\boldsymbol{\theta_j}$'s are the **oscillator phases.** One can also define oscillator frequencies, even when the θ_j's do not have the form $\theta_j = \omega_j\, t$, with ω_j constant. The idea is that, near any time t_0 we can write $\theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0)\,(t - t_0) + \ldots$, identifying $\dot{\theta}_j(t_0)$ as the local frequency. Hence, we define the **oscillator frequencies by $\boldsymbol{\tilde{\omega}_j = \dot{\theta}_j}$.** These frequencies are, of course, **generally not constants.**
5. The notion of phases can survive <u>even if</u> the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for θ_1 and θ_2 have an attracting critical point. We will see examples where this happens in the problems, e.g.: "Bifurcations in the torus # 01".

4.2 Phase locking and oscillator death

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system **phase locks.** This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.
2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is **oscillator death.** It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

THE END.