18.433 Combinatorial Optimization

Linear Programs

Oct 14 | *Lecturer: Santosh Vempala*

A linear program consists of linear constraints with the goal of maximizing or minimizing a linear objective function subject to the constraints.

Lets look at the max-flow problem:

$$G = (V, E)$$

In this problem the capacities are u_{ij} and the flows are x_{ij}. The conditions a flow must satisfy are:

$$0 \le x_{ij} \le u_{ij} \qquad \forall\ i, j \in E$$
$$\sum_j x_{ij} = \sum_j x_{ji} \qquad \forall\ v \in V \setminus \{s, t\}$$

Note that all of our constraints are linear as they are of the form:

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \gtreqless b$$

We would like to find the maximum of $\sum_j x_{sj} - \sum_j x_{js}$.

Let's look at a minimum cost flow problem. The constraints are:

$$0 \le x_{ij} \le u_{ij},$$
$$\sum_j x_{ij} - \sum_j x_{ji} = b(i) \qquad \forall\ i \in V.$$

The objective function is:

$$\sum_{i,j \in E} c_{ij} x_{ij}.$$

The goal is to minimize the objective function.

A maximum matching problem would have the following constraints:

$$0 \le x_e \le 1 \qquad \forall\ e \in E$$
$$\sum_{e:e \text{ meets } v} x_e \le 1 \qquad \forall\ v \in V$$
$$\sum_{e \in S} x_e \le \frac{|s| - 1}{2} \qquad \forall\ s \subseteq V,\ |s| \text{ odd}$$

The general form of a linear program is:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \qquad x, c \in \Re$$

$$\begin{array}{ccc} \text{Max } c^T x = \sum_{i=1}^{n} c_i x_i & \text{or} & \text{Min } c^T x = \text{Max } - c^T x \\ Ax \leq b, \; x \geq 0 & & a^T x \geq b \iff -a^T x \leq -b \end{array}$$

To solve:

$$\max cxAx \leq b$$

we can set $x = y - z$, where $y, z \geq 0$.

(Note: $a^T x \geq b \iff -a^T \leq b$)

We are trying to find an x such that the objective function is maximized. We must ask ourselves if there is a good characterization for the solution. Suppose we are given x^*. Is x^* the optimal solution?

If NO: Either x^* does not satisfy some constraint or give x^{**} such that

$c^T x^{**} > c^T x^*$.

If YES: ?? (We'll come back to this later)

Here is another example: Find the maximum value of $(4x_1 + x_2 + 5x_3 + 3x_4)$, call it z^*, subject to the following constraints:

$$x_1 - x_2 - x_3 + 3x_4 \leq 1 \tag{1}$$

$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \tag{2}$$

$$-x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \tag{3}$$

$$x_1, x_2, x_3, x_4 \geq 0 \tag{4}$$

Let's try to estimate z^* with a hit and miss method:

$$\begin{array}{ll} x = (0, 0, 1, 0) & z^* \geq 5 \\ x = (2, 1, 1, \frac{1}{3}) & z^* \geq 15 \end{array}$$

The problem with this method is that we don't know when z^* is a maximum. We need to find an upper bound on the optimum.

Let's try another approach: Equation 2 multiplied by $\frac{5}{3}$ is

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

Notice that the left side of this equation is term-by-term greater than or equal to the objective function. Therefore,

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{275}{3}.$$

And therefore,

$$z^* \leq \frac{275}{3}.$$

An even stricter bound can be obtained by adding Equations 2 and 3. This gives,

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$$

Again, this is term-by-term greater than or equal to the objective function, so,

$$z^* \leq 58.$$

Let us generalize this approach. Choose $y_1, y_2, y_3 \geq 0$ to be three multipliers on Equations 1, 2, and 3. Taking the sum we get:

$$\begin{aligned}(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + \\ + (3y_1 + 8y_2 - 5y_3)x_4 \geq y_1 + 55y_2 + 3y_3.\end{aligned} \tag{5}$$

In order for the left hand side of Equation 5 to be an upper bound on the objective function we require:

$$\left.\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}\right\} \Longrightarrow z^* \leq y_1 + 55y_2 + 3y_3$$

Therefore, in order to get the best upper bound we should minimize $(y_1 + 55y_2 + 3y_3)$ according to the above constraints. This constitutes a new linear program.

In general:

$$\left.\begin{array}{r} \max\ c^T x \\ Ax \le b \\ x_i \ge 0 \quad \forall\ i \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \max\ c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{array}\right.$$

Again, we would choose multipliers $y_1, y_2, \ldots, y_m \ge 0$ on the m constraint equations above.

The dual is:

$$\begin{gathered} \min\ b_1x_1 + b_2x_2 + \ldots + b_my_m \\ a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2 \\ \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n \\ y_i \ge 0 \quad \forall\ i \end{gathered}$$

Summarizing,

$$\underbrace{\begin{array}{c} \max\ c^T x \\ Ax \le b \\ x \ge 0 \end{array}}_{\text{Primal}} \qquad\qquad \underbrace{\begin{array}{c} \min\ b^T y \\ A^T y \ge c \\ y \ge 0 \end{array}}_{\text{Dual}}$$

(Note: Dual(Dual) = Primal)

Also,

$$\max\ c^T x \le \min\ b^T y,$$

therefore,

$$\begin{gathered} c^T x \le (A^T y)^T x = y^T A x \le y^T b = b^T y \\ \Rightarrow c^T x \le b^T y. \end{gathered}$$

This gives us the *Weak Duality Theorem*:

$$\max\ \{c^T x\ /\ Ax \le b\} \le \min\ \{y^T b\ /\ A^T y = c,\ y \ge 0\} \tag{6}$$

Next week we will prove the *Strong Duality Theorem* which replaces the inequality in Equation 6 with an equality. Using this we will be able to give a short proof of the case when x^* is optimal (i.e. the YES case mentioned earlier), which means that we have good characterization.