**Problem 1.** (Universality of Fredkin Gate)

See page 157.

Problem 2. Construct a binary adder using Toffoli gates (see page 29, Figure 1.14 for the Toffoli gate). This adder accepts two bits X and Y, and at the output, there is a carry bit as well as (X+Y mod 2).

The carry bit is one iff X and Y are both one, therefore it is equal to XY (hereafter, by multiplication we mean AND, and by "+", we mean addition mod 2.).

A Toffoli gate operates as follows:
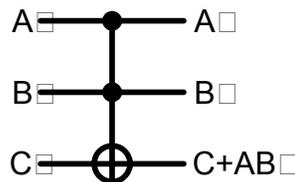
So, setting C=0, it operates as an AND gate. For A=1, the third output is C+B which is an XOR gate. So, the adder can be built as follows:
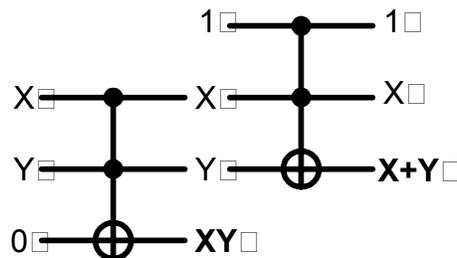
Extra Credit: Construct such an adder assuming there is also a carry-in bit Z at the input.

The carry bit in this case is XY+YZ+ZX=XY+Z(X+Y), which is one iff two or more of X, Y, and Z are one. The main output of the adder is X+Y+Z. So, we can use the following circuit in which the adder block refers to the adder in the previous part:
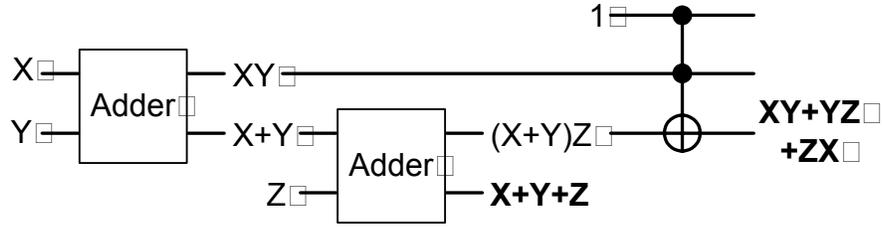
Problem 3.

Pauli matrices in the basis of $\{|0\rangle, |1\rangle\}$ can be represented as follows:

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Using the following eigenstate-eigenvalue relations:

$$\sigma_Z |0\rangle = |0\rangle, \sigma_Z |1\rangle = -|1\rangle$$
$$\sigma_X |+\rangle = |+\rangle, \sigma_X |-\rangle = -|-\rangle$$
$$\sigma_Y |\otimes\rangle = |\otimes\rangle, \sigma_Y |\odot\rangle = -|\odot\rangle$$

one can show that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, |\otimes\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, |\odot\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Now, each of the equations in the problem can be verified by matrix manipulations. To understand the rotation operation mentioned in part (a), we can use the Bloch sphere to geometrically visualize a single qubit (see page 15). In the Bloch sphere, a general qubit

$$|\psi\rangle = \cos(\theta / 2)|0\rangle + e^{i\phi} \sin(\theta / 2)|1\rangle$$

can be represented by a unit-length vector at the spherical coordinate $(1, \phi, \theta)$. Here, we have ignored the local phase factor. It can be verified that vectors $|0\rangle$ and $|1\rangle$, respectively, point to the positive and negative directions of the z-axis. The same holds for $|+\rangle$ and $|-\rangle$ with regard to the x-axis, as well as $|\otimes\rangle$ and $|\odot\rangle$ with regard to the y-axis. Hence, in order to show that σ_X is equivalent to a rotation by π radians around x-axis, it suffices to show that σ_X takes $|+\rangle$ and $|-\rangle$ to themselves, but takes $|\otimes\rangle$ to $|\odot\rangle$ and $|0\rangle$ to $|1\rangle$ (in all cases, including a possible local phase, which does not appear in the Bloch sphere representation).