Section 9

Most powerful tests for two simple hypotheses.

Now that we learned how to construct the decision rule that minimizes the Bayes error we will turn to our next goal - constructing the decision rule with controlled error of type 1 that minimizes error of type 2. Given $\alpha \in [0, 1]$ we consider the class of decision rules

$$K_\alpha = \{\delta : \mathbb{P}_1(\delta \neq H_1) \leq \alpha\}$$

and we will try to find $\delta \in K_\alpha$ that makes the type 2 error $\alpha_2 = \mathbb{P}_2(\delta \neq H_2)$ as small as possible.

Theorem. *Assume that there exists a constant c, such that*

$$\mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} < c\Big) = \alpha. \tag{9.0.1}$$

Then the decision rule

$$\delta = \begin{cases} H_1 : & \frac{f_1(X)}{f_2(X)} \geq c \\ H_2 : & \frac{f_1(X)}{f_2(X)} < c \end{cases} \tag{9.0.2}$$

is the most powerful in class K_α.

Proof. Take $\xi(1)$ and $\xi(2)$ such that

$$\xi(1) + \xi(2) = 1, \quad \frac{\xi(2)}{\xi(1)} = c,$$

i.e.

$$\xi(1) = \frac{1}{1+c} \quad \text{and} \quad \xi(2) = \frac{c}{1+c}.$$

Then the decision rule δ in (9.0.2) is the Bayes decision rule corresponding to weights $\xi(1)$ and $\xi(2)$ which can be seen by comparing it with (8.0.1), only here we break the tie in favor of H_1. Therefore, this decision rule δ minimizes the Bayes error which means that for any other decision rule δ',

$$\xi(1)\mathbb{P}_1(\delta \neq H_1) + \xi(2)\mathbb{P}_2(\delta \neq H_2) \leq \xi(1)\mathbb{P}_1(\delta' \neq H_1) + \xi(2)\mathbb{P}_2(\delta' \neq H_2). \tag{9.0.3}$$

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By assumption (9.0.1) we have

$$\mathbb{P}_1(\delta \neq H_1) = \mathbb{P}_1\left(\frac{f_1(X)}{f_2(X)} < c\right) = \alpha,$$

which means that δ comes from the class K_α. If $\delta' \in K_\alpha$ then

$$\mathbb{P}_1(\delta' \neq H_1) \leq \alpha$$

and equation (9.0.3) gives us that

$$\xi(1)\alpha + \xi(2)\mathbb{P}_2(\delta \neq H_2) \leq \xi(1)\alpha + \xi(2)\mathbb{P}_2(\delta' \neq H_2)$$

and, therefore,
$$\mathbb{P}_2(\delta \neq H_2) \leq \mathbb{P}_2(\delta' \neq H_2).$$

This exactly means that δ is more powerful than any other decision rule in class K_α.

$\square$

Example. Suppose we have a sample $X = (X_1, \ldots, X_n)$ and two simple hypotheses $H_1 : \mathbb{P} = N(0,1)$ and $H_2 : \mathbb{P} = N(1,1)$. Let us find most powerful δ with the error of type 1

$$\alpha_1 \leq \alpha = 0.05.$$

According to the above Theorem if we can find c such that

$$\mathbb{P}_1\left(\frac{f_1(X)}{f_2(X)} < c\right) = \alpha = 0.05$$

then we know how to find δ. Simplifying this equation gives

$$\mathbb{P}_1\left(\sum X_i > \frac{n}{2} - \log c\right) = \alpha = 0.05$$

or
$$\mathbb{P}_1\left(\frac{1}{\sqrt{n}}\sum X_i > c' = \frac{1}{\sqrt{n}}\left(\frac{n}{2} - \log c\right)\right) = \alpha = 0.05.$$

But under the hypothesis H_1 the sample comes from standard normal distribution $\mathbb{P}_1 = N(0,1)$ which implies that the random variable

$$Y = \frac{1}{\sqrt{n}}\sum X_i$$

is standard normal. We can use the table of normal distribution to find

$$\mathbb{P}(Y > c') = \alpha = 0.05 \Rightarrow c' = 1.64.$$

Therefore, the most powerful test with level of significance $\alpha = 0.05$ is:

$$\delta = \begin{cases} H_1 : & \frac{1}{\sqrt{n}}\sum X_i \leq 1.64 \\ H_2 : & \frac{1}{\sqrt{n}}\sum X_i > 1.64. \end{cases}$$

What will the error of type 2 be for this test?

$$\begin{aligned} \alpha_2 &= \mathbb{P}_2(\delta \neq H_2) = \mathbb{P}_2\Big(\frac{1}{\sqrt{n}}\sum X_i \leq 1.64\Big) \\ &= \mathbb{P}_2\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - 1) \leq 1.64 - \sqrt{n}\Big). \end{aligned}$$

The reason we subtracted 1 from each X_i is because under the second hypothesis X's have distribution $N(1,1)$ and random variable

$$Y = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - 1)$$

will be standard normal. Therefore, the error of type 2 for this test will be equal to

$$\mathbb{P}(Y < 1.64 - \sqrt{n}) = N(0,1)(-\infty, 1.64 - \sqrt{n}).$$

For example, when the sample size $n = 10$, $\alpha_2 = \mathbb{P}(Y < 1.64 - \sqrt{10}) = 0.087$.

□

Randomized most powerful test.

Next we will show how to get rid of the assumption (9.0.1) which, unfortunately, does not always hold as will become clear from examples below.

If we examine carefully the proof of Theorem we notice that condition (9.0.1) was necessary to ensure that the likelihood ratio test has error of type 1 *exactly*! equal to α. Also, the test was designed to be a Bayes test and in Bayes tests we have a freedom of breaking a tie in an arbitrary way. In the following version of previous theorem we will show that the most powerful test in class K_α can always be constructed by breaking a tie randomly in a way that makes error of type 1 exactly equal to α.

Theorem. *Given any $\alpha \in [0,1]$ we can always find $c \in [0,\infty)$ and $p \in [0,1]$ such that*

$$\mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} < c\Big) + (1-p)\mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} = c\Big) = \alpha. \tag{9.0.4}$$

In this case, the most powerful test $\delta \in K_\alpha$ is given by

$$\delta = \begin{cases} H_1 : & \frac{f_1(X)}{f_2(X)} > c \\ H_2 : & \frac{f_1(X)}{f_2(X)} < c \\ pH_1 + (1-p)H_2 : & \frac{f_1(X)}{f_2(X)} = c, \end{cases}$$

where in the case of equality we break the tie randomly by picking H_1 with probability p and H_2 with probability $1-p$. This test δ is called a randomized most powerful test for two simple hypotheses at the level of significance α.

Proof. Let us first assume that we can find c and p such that (9.0.4) holds. Then the error of type 1 for the randomized test δ is

$$\alpha_1 = \mathbb{P}_1(\delta \neq H_1) = \mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} < c\Big) + (1-p)\mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} = c\Big) = \alpha, \tag{9.0.5}$$

since δ does not pick H_1 exactly when the likelihood ratio is less than c or when it is equal to c in which case H_1 is not picked with probability $1 - p$. This means that the randomized test $\delta \in K_\alpha$. The rest of the proof repeats the proof of the previous theorem. We only need to point out that our randomized test will still be a Bayes test since in the case of equality

$$\frac{f_1(X)}{f_2(X)} = c$$

the Bayes test allows us to break the tie arbitrarily and we choose to break it randomly in a way that ensures that the error of type one will be equal to α, as in (9.0.5).

The only question left is why we can always choose c and p such that (9.0.4) is satisfied. If we look at the function

$$F(t) = \mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} < t\Big)$$

as a function of t, it will increase from 0 to 1 as t increases from 0 to ∞. Let us keep in mind that, in general, $F(t)$ might have jumps. We can have two possibilities: either (a) at some point $t = c$ the function $F(c)$ will be equal to α, i.e.

$$F(c) = \mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} < c\Big) = \alpha$$

or (b) at some point $t = c$ it will jump over α, i.e.

$$F(c) = \mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} < c\Big) < \alpha$$

but

$$\mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} \leq c\Big) = F(c) + \mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} = c\Big) \geq \alpha.$$

In both cases we find p by solving (9.0.4) with this value of c. In the case (a) we get $p = 1$ and for (b) we get

$$1 - p = (\alpha - F(c)) \Big/ \mathbb{P}\Big(\frac{f_1(X)}{f_2(X)} = c\Big).$$

$\square$

Example. Suppose that we have one observation $X \sim B(p)$ from Bernoulli distribution with probability of success $p \in (0, 1)$. Consider two simple hypotheses

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\begin{aligned}
H_1 : p = 0.2, \text{ i.e. } f_1(x) &= 0.2^x 0.8^{1-x}, \\
H_2 : p = 0.4, \text{ i.e. } f_2(x) &= 0.4^x 0.6^{1-x}.
\end{aligned}
$$

Let us take the level of significance $\alpha = 0.05$ and find the most powerful $\delta \in K_{0.05}$. In figure 9.1 we show the graph of the function

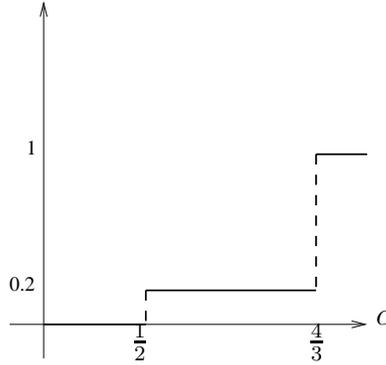$$F(c) = \mathbb{P}_1\Big(\frac{f_1(X)}{f_2(X)} < c\Big).$$

Figure 9.1: Graph of $F(c)$.

Let us explain how this graph is obtained. First of all, the likelihood ratio can take only two values:
$$\frac{f_1(X)}{f_2(X)} = \begin{cases} 1/2, & \text{if} \quad X = 1 \\ 4/3, & \text{if} \quad X = 0. \end{cases}$$

If $c \leq 1/2$ then the set
$$\left\{ \frac{f_1(X)}{f_2(X)} < c \right\} = \emptyset \text{ is empty and } F(c) = \mathbb{P}_1(\emptyset) = 0.$$

If $1/2 < c \leq 4/3$ then the set
$$\left\{ \frac{f_1(X)}{f_2(X)} < c \right\} = \{X = 1\} \text{ and } F(c) = \mathbb{P}_1(X = 1) = 0.2.$$

Finally, if $4/3 < c$ then the set
$$\left\{ \frac{f_1(X)}{f_2(X)} < c \right\} = \{X = 0 \text{ or } 1\} \text{ and } F(c) = \mathbb{P}_1(X = 0 \text{ or } 1) = 1.$$

The function $F(c)$ jumps over the level $\alpha = 0.05$ at the point $c = 1/2$. To determine p we have to make sure that the error of type one is equal to 0.05, i.e.
$$\mathbb{P}_1\left(\frac{f_1(X)}{f_2(X)} < c\right) + (1-p)\mathbb{P}_1\left(\frac{f_1(X)}{f_2(X)} = c\right) = 0 + (1-p)0.2 = 0.05$$

which gives that $p = 3/4$. Therefore, the most powerful test of size $\alpha = 0.05$ is
$$\delta = \begin{cases} H_1: & \frac{f_1(X)}{f_2(X)} > \frac{1}{2} \text{ or } X = 0 \\ H_2: & \frac{f_1(X)}{f_2(X)} < \frac{1}{2} \text{ or never} \\ \frac{3}{4}H_1 + \frac{1}{4}H_2: & \frac{f_1(X)}{f_2(X)} = \frac{1}{2} \text{ or } X = 1. \end{cases}$$

$\square$

In the example above one could also consider two or more observations. Another example would be to consider, let's say, two observations from Poisson distribution $\Pi(\lambda)$ and test two simple hypotheses $H_1 : \lambda = 0.1$ vs. $H_2 : \lambda = 0.3$.

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