NAME: ________________________________

Spring 2018 18.600 Final Exam: 100 points

Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the unit circle $\{(x, y) : x^2 + y^2 \leq 1\}$.

(a) Give the joint probability density function $f(x, y)$ for the pair (X, Y).

(b) Compute the marginal law $f_X(x)$.

(c) Compute the conditional expectation $E[X^2|Y]$ as a function of Y.

2. (10 points) Ivan is waiting for the bus. There are three kinds of buses, whose arrival times are three independent Poisson point processes. During each hour, on average one expects to see 3 yellow buses, 2 blue buses, and 1 red bus at the bus stop where Ivan is waiting.

(a) Give the probability density function for T where T is the length of time until the first bus of *any* kind shows up.

(b) Compute the probability that there are exactly two yellow buses during the first half hour.

(c) Suppose that a yellow bus would get Ivan home in 30 minutes (from the time it arrives at the bus stop until the time it gets to his house) while either a red or a blue bus would get Ivan home in 15 minutes (from the time it picks Ivan up at the bus stop until the time it gets to his house). If Ivan wants to minimize the expected amount of time until he gets home, and a yellow bus arrives first, should he get on the yellow bus or should he wait for a red/blue bus to come? Give a sentence or two of explanation.

3. (10 points) Harriet and Helen are real estate agents. Each week, each agent is sent to close a deal, and one of the two agents closes a deal 10 percent of the time, while the other closes it 20 percent of the time. To set notation, let X_j be 1 if the more capable agent closes the deal the jth week and 0 otherwise. Let Y_j be 1 if the less capable agent closes the deal on the jth week, and 0 otherwise. So each X_j is equal to 1 with probability 2/10 and 0 with probability 8/10. Each Y_j is equal to 1 with probability 1/10 and 0 with probability 9/10. Assume that the random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are all independent of each other. For each $j \geq 1$ write $Z_j = X_j - Y_j$. Write $S_n = \sum_{j=1}^n Z_j$.

(a) Compute the mean, variance and standard deviation of Z_1.

(b) Compute the mean, variance and standard deviation of S_{100}.

(c) Use the central limit theorem to approximate the probability that $S_{100} > 0$. (This is the probability that the more capable closer has closed more deals after 100 weeks.) You may use the function $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.

4. (10 points) A future society has exactly seven software companies. (You can guess what they are.) I will call them Company 1, Company 2, Company 3, Company 4, Company 5, Company 6, and Company 7. Every software engineer either works for one of these companies or is unemployed. At the end of each month, a survey asks what each software engineer is doing. It happens that...

> A software engineer who is unemployed one month has a 3/10 chance to be unemployed the next month, and (for each $j \in \{1, 2, \ldots, 7\}$) a 1/10 chance to be at Company j.
>
> A software engineer at Company j one month (for any $j \in \{1, 2, \ldots, 7\}$) has a 69/70 chance to remain at Company j the next month, and a 1/70 chance to be unemployed.

Complete the following:

(a) Represent the employment activity as a Markov chain by writing down an 8 by 8 transition matrix with states numbered 0 (unemployment) or 1 to 7 (the companies). You don't have to write out all 64 terms. Just write the non-zero terms.

(b) If a software engineer is currently unemployed, what is the probability that he or she will be unemployed in two months?

(c) Compute $(\pi_0, \pi_1, \pi_2, \ldots, \pi_7)$ where π_i is the fraction of months spent in state i over the long term. [Hint: start by working out π_0.]

5. (10 points) Each of the customers calling a certain toaster customer service number fits into one of seven predictable categories:

1/4 want to purchase a new toaster.

1/4 want to complain about toast being too dark and crispy even on light settings.

1/8 want to complain about a toaster that is not connecting to the internet properly.

1/8 want to complain about a hand being stuck in a toaster.

1/8 want to complain about a toaster having started a fire.

1/16 want to know how to make cinnamon toast.

1/16 want to know how to adjust the springs so that the toast really flies into the air.

Let X be the category of the next customer who will call. We do not yet know X so we model it as a "category-valued" random variable, which takes one of the above seven values with the indicated probabilities.

(a) Compute the entropy $H(X)$.

(b) Imagine that the call center determines the category by asking the customer a sequence of yes or no questions, using a strategy that minimizes the expected number of questions asked. What is the expected number of questions asked?

(c) Suppose $X_1, X_2, X_3, \ldots, X_{100}$ are i.i.d. random variables, each with the same law as X, representing the calls received over an entire day. Compute $H(X_1, X_2, X_3, \ldots, X_{100})$.

(d) Let Y be the total number of callers during the day who want to purchase a new toaster. So Y is a random integer between 0 and 100. Write $Z = Y^2$. Compute the entropy $H(Z)$. You can write this as a sum; no need to compute an explicit number.

6. (10 points) Let $X_1, X_2, \ldots$ be i.i.d. random variables, each with density function $\frac{1}{\pi(1+x^2)}$.

(a) Write $S_0 = 0$ and (for $n \geq 1$) write $S_n = \sum_{j=1}^{n} X_j$. Is the sequence S_n a martingale? Explain why or why not. (Recall that the definition of martingale requires that $E[\ S_n|] < \infty$ for all n, and explicitly note whether this is true.)

(b) Compute the probability density function for S_{10}.

(c) Compute the probability $P(S_{10} > 10)$.

7. (10 points) Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. random variables, each equal to -2 with probability $1/2$ and 2 with probability $1/2$. Write $S_0 = 0$ and (for $n \geq 1$) write $S_n = \sum_{j=1}^{n} X_j$.

(a) Compute the probability that the sequence $S_0, S_1, S_2, \ldots$ reaches 20 before the first time that it reaches -30.

(b) Compute the probability that there exists *some* positive integer n for which $S_n = -20$.

(c) Compute the moment generating function $M_{X_1}(t)$.

(d) Compute the moment generating function $M_{S_9}(t)$.

8. (10 points) Suppose there are 6 cards with labels $1, 2, 3, 4, 5, 6$. The cards are shuffled to a uniformly random permutation and are then turned over one at a time. Each time one sees a card that is higher in numerical value than all other preceding cards, one excitedly shouts "Record!" (The first card to be turned over is automatically a record, since it has no competition.) For example, if the cards are turned over in the order $1, 2, 3, 4, 5, 6$ then one will shout "Record!" all 6 times, since each card has a larger number than all previous numbers. If the cards are turned over in the order $6, 3, 4, 5, 1, 2$ then one will shout "Record!" only once. Let R be the total number of times "Record!" is shouted. (So R is an integer-valued random variable taking values between 1 and 6.) To further set notation, let A_i be 1 if the ith card turned over is a record and 0 otherwise. Compute the following:

(a) The expectation $E[R]$.

(b) The variance $\mathrm{Var}[R]$.

(c) Compute the conditional probability that the last three cards turned over are all records *given* that the first three cards are all records.

9. (10 points) Let $X_1, X_2, X_3 \dots$ be an infinite sequence of i.i.d. normal random variables, each with mean zero and variance one.

(a) Is the sequence $X_1, X_2, \dots$ a martingale? Explain why or why not in one sentence.

(b) Compute the correlation coefficient of $A = \sum_{n=1}^{60} X_n$ and $B = \sum_{n=41}^{100} X_n$.

(c) Write down the probability density function for A.

(d) Compute the variance of $A + B$.

10. (10 points) Let X, Y, Z be i.i.d. exponential random variables, each with parameter $\lambda = 1$.

(a) Compute the probability density function for $X + Y + Z$.

(b) Compute the expectation and variance of $\max\{X, Y, Z\}$.

(c) Compute the expectation and variance of $\min\{X, Y, Z\}$.

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