18.440 Midterm 1, Fall 2012: Solutions

1. (20 points) Toss 6 independent fair coins, and let X be the number that come up heads. Compute the following:

   (a) $E[X]$ **ANSWER:** X is binomial $(n, p) = (6, 1/2)$ so $E[X] = np = 3$

   (b) $\text{Var}[X]$ **ANSWER:** $np(1 - p) = 1.5$

   (c) $P[X = 6 | X \geq 5]$ **ANSWER:**
   $P[X = 6]/P[X \geq 5] = 2^{-6}/(2^{-6} + \binom{6}{5}2^{-6}) = 1/7$.

2. (20 points) Four people have two hats each. They take off these hats and shuffle them up randomly and redistribute them among the four people (so that each again has two hats). Assume that all the ways of dividing the 8 hats among the 4 people (with each person getting two hats) are equally likely.

   (a) How many possible outcomes are there (i.e., how many ways are there to divide 8 distinct hats among four distinct people, with each person getting two hats)? **ANSWER:** $\binom{8}{2\,2\,2\,2} = 8!/16$

   (b) Label the people 1, 2, 3, and 4. Let E_i be the event that the ith person gets *both* of his or her own hats. Compute $P[E_1]$.
   ANSWER: $\frac{1}{8} \cdot \frac{1}{7} \cdot 2 = \frac{1}{28}$

   (c) Compute $P[E_1 E_2]$. Then compute $P[E_1 E_2 E_3]$ (which is the same as $P[E_1 E_2 E_3 E_4]$). **ANSWER:**

   $$P[E_1 E_2] = \frac{1}{8} \cdot \frac{1}{7} \cdot 2 \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot 2 = 4/(8 \cdot 7 \cdot 6 \cdot 5) = 1/420$$

   $$P[E_1 E_2 E_3] = P[E_1 E_2 E_3 E_4] = 16/8!$$

   (d) Write $A = P[E_1]$, $B = P[E_1 E_2]$, $C = P[E_1 E_2 E_3] = P[E_1 E_2 E_3 E_4]$. Use inclusion-exclusion to express $P[E_1 \cup E_2 \cup E_3 \cup E_4]$ in terms of A, B, and C. **ANSWER:** $4A - 6B + 4C - C$

3. (20 points) Suppose that during each minute of a 90-minute soccer game there is a probability of 2/90 that one goal will be scored and a probability of 88/90 that no goal will be scored (independently of all other minutes). Let N be the total number of goals scored during the game.

(a) Compute $E[N]$ and $\text{Var}[N]$. (Give exact answers, not approximate ones.) **ANSWER:** $E[N] = 90 \cdot \frac{2}{90} = 2$ and $\text{Var}[N] = 90 \cdot \frac{2}{90} \cdot \frac{88}{90} = \frac{176}{90}$

(b) Compute the probability that there is exactly one goal. Give an *exact* answer. **ANSWER:** $\binom{90}{1}(2/90)^1(88/90)^{89}$

(c) Let E be the event that there are exactly 0 goals in the first half and exactly 2 goals in the second half. Use a Poisson random variable calculation to *approximate* the probability of E. **ANSWER:** Number in each half is approximately Poison(λ) with $\lambda = 1$. So $P(E) \approx e^{-\lambda}\lambda^1/1! \cdot e^{-\lambda}\lambda^2/2! = \frac{1}{2e^{-2}}$

4. (10 points) Roll 6 independent fair six-sided dice (so that each of the 6^6 outcomes is equally likely). Compute the probability that exactly three of the dice land on even numbers. **ANSWER:** $\binom{6}{3}2^{-6} = 20/64 = 5/16$.

5. (20 points) Roll 2 independent fair six-sided dice. Let X be the value on the first die, Y the value on the second die. Write $Z = Y + X$. Compute the following:

(a) $E[Z]$. **ANSWER:** $E[Z] = E[X] + E[Y] = 7$

(b) $E[Y^2]$. **ANSWER:** $E[Y^2] = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$

(c) $E[YZ]$. **ANSWER:** $[YZ] = E[Y^2 + XY] = (91/6) + (7/2)^2$

6. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability 1/3. Let X be such that the third heads occurs on the Xth toss.

(a) Compute $P[X = 9]$. **ANSWER:** $\binom{8}{2}(1/3)^2(2/3)^6(1/3)$

(b) Compute $E[X]$. **ANSWER:** $3/(1/3) = 9$

18.600 Probability and Random Variables
Fall 2019