**18.600 Midterm 1, Spring 2019: 50 minutes, 100 points**

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).**

NAME: ________________________________________

1. (20 points) A town has 2000 residents. An obscure film is playing in its only theater. Each resident decides independently whether to view the film, and each resident views the film with probability 1/1000. Let $X$ be the number of people who view the film.

(a) Compute $E[X]$. Given an exact answer, not an approximation.

(b) Compute $\mathrm{Var}[X]$. Give an exact answer, not an approximation.

(c) Compute $E[X^2]$. Give an exact answer, not an approximation.

(d) Use a Poisson random variable to *approximate* $P(X = 4)$.

2.(10 points) Suppose that $X$ is a Poisson random variable with parameter 2 and $Y$ is a Poisson random variable with parameter 3.

(a) Compute the expectation $E(3X + 4Y + 5)$.

(b) Compute the variance $\mathrm{Var}(5X + 7)$.

3. (20 points) Alice, Bob, Carol, Dave, Eve, and Frank are gathered together for a night of pizza and dungeons and dragons. They order two large pizzas, each cut into 12 pieces, so there are 24 pieces altogether.

(a) How many ways are there to divide the 24 (indistinguishable) pieces among the six people? in other words, how many sequences $a_1, a_2, \ldots, a_6$ of *non-negative* integers satisfy $\sum_{i=1}^{6} a_i = 24$?

(b) Eve proposes that, for the sake of fairness, only divisions in which each person gets *at least one slice* of pizza should be considered. How many sequences $a_1, a_2, \ldots, a_6$ of *strictly positive* integers satisfy $\sum_{i=1}^{6} a_i = 24$?

(c) Each of the six players pulls out a fair twenty-sided die (containing the numbers $\{1, 2, \ldots, 20\}$) and rolls it. (The six rolls are independent of each other.) What is the probability that the sum of the numbers on the dice is exactly 24?

4. (20 points) An a capella group with 15 members (8 women and 7 men) is organizing a holiday gift exchange. Each member writes his or her name on a piece of paper and puts it in a bowl. Then the pieces of paper are randomly distributed among the 15 people, with all 15! arrangements being equally likely. Each person is assigned to buy a gift for the individual on the paper that he or she chose.

(a) Compute the expected number of people who will be assigned to buy gifts for themselves.

(b) Compute the expected number of men who will be assigned to give gifts to women.

(c) Compute the probability that *every* man is assigned to give a gift to a woman.

(d) Compute the probability that *every* individual is part of a cycle of length three (i.e., a group of people $A$, $B$, and $C$ where $A$ gives to $B$, $B$ gives to $C$, and $C$ gives to $A$).

5. (10 points) A standard deck of 52 cards has 13 cards of each suit (diamonds, hearts, clubs, or spades). The deck is randomly divided into 4 bridge hands with 13 cards each (with all divisions being equally likely). What is the probability that *each* of these hands contains cards from only a single suit? (So one hand is only hearts, one hand is only clubs, and so forth.)

6. (20 points) Alicia is writing a paper for her history class. Whenever she writes a paper, there is a .7 chance it will be brilliant and a .3 chance it will be mediocre. A professor reading a brilliant paper gives it an A with probability .9. A professor reading a mediocre paper gives it an $A$ with probability .3. Let $B$ be the event that that the paper is brilliant and let $A$ be the event that it gets an A grade, so that our assumptions can be stated as $P(B) = .7$ and $P(A|B) = .9$ and $P(A|B^c) = .3$. Now compute the following:

(a) $P(A)$ (i.e., overall likelihood she gets an $A$)

(b) $P(B|A)$ (i.e., likelihood paper is brilliant given it got an $A$)

(c) $P(B|A^c)$ (i.e., likelihood paper is brilliant given it did not get an $A$)