18.440 Midterm 2, Fall 2012: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (10 points)
Suppose that a fair die is rolled 18000 times. Each roll turns up a uniformly random member of the set $\{1, 2, 3, 4, 5, 6\}$ and the rolls are independent of each other. Let X be the total number of times the die comes up 1.

(a) Compute the variance $\mathrm{Var}(X)$.

(b) Use a normal random variable approximation to estimate the probability that $X < 2900$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer.

2. (20 points) Let X_1, X_2, and X_3 be independent uniform random variables on $[0, 1]$. Write $Y = X_1 + X_2$ and $Z = X_2 + X_3$.

(a) Compute $E[X_1 X_2 X_3]$.

(b) Compute the variance $\mathrm{Var}(X_1)$.

(c) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$.

(d) Compute and draw a graph of the density function f_Y.

3. (20 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on $[0, 1]$.

(a) Write $Y = \min\{X_1, X_2, \ldots, X_n\}$. Compute the cumulative distribution function $F_Y(a)$ and the density function $f_Y(a)$ for $a \in [0, 1]$.

(b) Compute $P(X_1 < .3)$ and $P\big(\max\{X_1, X_2, \ldots, X_n\}\big) < .3$.

(c) Compute the expectation $E[X_1 + X_2 + \ldots + X_n]$.

4. (20 points) Aspiring writer Rachel decides to lock herself in her room to think of screenplay ideas. When Rachel is thinking, the moments at which good new ideas occur to her form a Poisson process with parameter $\lambda_G = .5/\text{hour}$. The times when bad new ideas occur to her are a Poisson point process with parameter $\lambda_B = 1.5$ per hour.

(a) Let T be the amount of time until Rachel has her first idea (good or bad). Write down the probability density function for T.

(b) Compute the probability that Rachel has exactly 3 bad ideas total during her first hour of thinking.

(c) Let S be the amount of time elapsed before the fifth good idea occurs. Compute $\text{Var}(S)$.

(d) What is the probability that Rachel has no ideas at all during her first three hours of thinking?

5. (20 points) Suppose that X and Y have a joint density function f given by

$$f(x, y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X.

(b) Compute the conditional expectation $E[X|Y = .5]$.

(c) Express $E[X^3Y^3]$ as a double integral. (You don't have to explicitly evaluate the integral.)

6. (10 points) Let X and Y be independent normal random variables, each with mean 1 and variance 9.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f.

(b) Compute $E[X^2]$ and $E[X^2Y^2]$.

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18.600 Probability and Random Variables
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