18.440 Midterm 2, Fall 2012: 50 minutes, 100 points

1. (10 points) Suppose that a fair die is rolled 18000 times. Each roll turns up a uniformly random member of the set $\{1, 2, 3, 4, 5, 6\}$ and the rolls are independent of each other. Let X be the total number of times the die comes up 1.

(a) Compute $\mathrm{Var}(X)$. **ANSWER:** $npq = 18000(5/6)(1/6) = 2500$

(b) Use a normal random variable approximation to estimate the probability that $X < 2900$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** Standard derivation is $\sqrt{2500} = 50$. Probability X more than 2 standard deviations below mean is approximately $\Phi(-2)$.

2. (20 points) Let X_1, X_2, and X_3 be independent uniform random variables on $[0, 1]$. Write $Y = X_1 + X_2$ and $Z = X_2 + X_3$.

(a) Compute $E[X_1X_2X_3]$. **ANSWER:** Independence implies $E[X_1X_2X_3] = E[X_1]E[X_2]E[X_3] = (1/2)^3 = 1/8$.

(b) Compute $\mathrm{Var}(X_1)$. **ANSWER:** $E(X_1^2) = \int_0^1 x^2 dx = 1/3$, so $\mathrm{Var}(X_1) = E(X_1^2) - E(X_1)^2 = 1/3 - 1/4 = 1/12$.

(c) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$. **ANSWER:** By bilinearity of covariance,

$$\mathrm{Cov}(Y, Z) = \mathrm{Cov}(X_1 + X_2, X_2 + X_3)$$

$$= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_2, X_2) + \mathrm{Cov}(X_2, X_3).$$

All terms are zero by independence except $\mathrm{Cov}(X_2, X_2) = \mathrm{Var}(X_2) = 1/12$. Then $\rho(Y, Z) = \frac{1/12}{\sqrt{(2/12)(2/12)}} = 1/2$.

(d) Compute and draw a graph of the density function f_Y. **ANSWER:** $f_Y(a) = \int_{-\infty}^{\infty} f_X(a-y) f_X(y) dy$ where

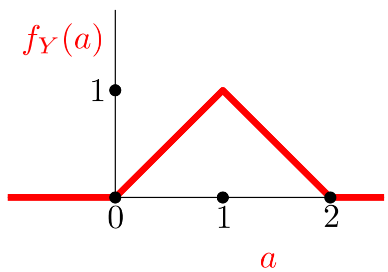
$$f_X(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}.$$

Then

$$f_X(a-y)f_X(y) = \begin{cases} 1 & a - y \in (0, 1), y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Now $a - y \in (0,1)$ is equivalent to $-y \in (-a, 1-a)$ or equivalently $y \in (a-1, a)$. Thus $f_Y(a)$ is equal to the length of the intersection of the intervals $(0,1)$ and $(a-1,a)$. This becomes

$$f_Y(a) = \begin{cases} 0 & a < 0 \\ a & 0 \le a < 1 \\ 2-a & 1 \le a < 2 \\ 0 a \ge 2 \end{cases}$$



3. (20 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on $[0,1]$.

(a) Write $Y = \min\{X_1, X_2, \ldots, X_n\}$. Compute the cumulative distribution function $F_Y(a)$ and the density function $f_Y(a)$ for $a \in [0,1]$. **ANSWER:** By independence, $P(\min\{X_1, X_2, \ldots, X_n\} > a) = P(X_1 > a)P(X_2 > a) \cdots P(X_n > a) = (1-a)^n$ So $F_Y(a) = 1 - (1-a)^n$, and $f_Y(a) = F_Y'(a) = n(1-a)^{n-1}$

(b) Compute $P(X_1 < .3)$ and $P\big(\max\{X_1, X_2, \ldots, X_n\}\big) < .3$. **ANSWER:** $.3$ and $.3^n$.

(c) Compute the expectation $E[X_1 + X_2 + \ldots + X_n]$. **ANSWER:** By additivity of expectation, this is $nE[X_1] = n/2$.

4. (20 points) Aspiring writer Rachel decides to lock herself in her room to think of screenplay ideas. When Rachel is thinking, the moments at which good new ideas occur to her form a Poisson process with parameter $\lambda_G = .5$/hour. The times when bad new ideas occur to her are a Poisson point process with parameter $\lambda_B = 1.5$ per hour.

(a) Let T be the amount of time until Rachel has her first idea (good or bad). Write down the probability density function for T.

ANSWER: T is exponential with parameter $\lambda = \lambda_G + \lambda_B = 2$, so $f_T(x) = 2e^{-2x}$.

(b) Compute the probability that Rachel has exactly 3 bad ideas total during her first hour of thinking. **ANSWER:** Number N of bad ideas is Poisson with rate $1 \cdot \lambda_B = 1.5$. So $P(N = 3) = \frac{(1.5)^3 e^{-1.5}}{3!}$.

(c) Let S be the amount of time elapsed before the fifth good idea occurs. Compute $\mathrm{Var}(S)$. **ANSWER:** Variance of time till one good idea is $1/\lambda_G^2$. Memoryless property and additivity of variance of independent sums gives $\mathrm{Var}(S) = 5/\lambda_G^2 = 20$.

(d) What is the probability that Rachel has no ideas at all during her first three hours of thinking? **ANSWER:** Time till first idea is exponential with $\lambda = 2$. Probability this time exceeds 3 is $e^{-2 \cdot 3} = e^{-6}$.

5. (20 points) Suppose that X and Y have a joint density function f given by

$$f(x,y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X. **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \frac{1}{\pi} 2\sqrt{1-x^2} & -1 \leq x \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute the conditional expectation $E[X|Y = .5]$. **ANSWER:** Probability density for X given $Y = .5$ is uniform on $(-\sqrt{1-.5^2}, \sqrt{1-.5^2})$. So $E[X|Y = .5] = 0$.

(c) Express $E[X^3Y^3]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:**

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} x^3 y^3 dy dx.$$

6. (10 points) Let X and Y be independent normal random variables, each with mean 1 and variance 9.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f. **ANSWER:**

$$f(x,y) = \frac{1}{3\sqrt{2\pi}} e^{-(x-1)^2/18} \frac{1}{3\sqrt{2\pi}} e^{-(y-1)^2/18} = \frac{1}{18\pi} e^{-\frac{(x-1)^2-(y-1)^2}{18}}.$$

(b) Compute $E[X^2]$ and $E[X^2Y^2]$. **ANSWER:**
$\mathrm{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - 1 = 9$, so $E[X^2] = 10$. By independence $E[X^2Y^2] = E[X^2]E[Y^2] = 100$.