18.600 Midterm 2, Spring 2017 Solutions

1. (10 points) Suppose that X, Y, and Z are independent random variables, each of which is equal to 1 with probability 1/3 and 5 with probability 2/3. Write $W = X + Y + Z$.

(a) Compute the moment generating function M_X. **ANSWER:** $M_X(t) = \frac{1}{3}e^t + \frac{2}{3}e^{5t}$

(b) Compute the moment generating function M_W. **ANSWER:** $M_W(t) = \left(\frac{1}{3}e^t + \frac{2}{3}e^{5t}\right)^3$

2. (15 points) Let $X_1, X_2, \ldots, X_7$ be independent normal random variables, each with mean 0 and variance 1. Write $Z = \sum_{j=1}^{7} X_j$.

(a) Give the probability density function for Z. **ANSWER:** Sum is normal with mean zero and variance $\sigma^2 = 7$ so

$$F_Z(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}} = \frac{1}{\sqrt{14\pi}}e^{\frac{-x^2}{14}}.$$

b) Compute the probability that that the random variables are in increasing order, i.e., that $X_1 < X_2 < X_3 < \ldots < X_7$. **ANSWER:** All orderings are equally likely by symmetry, so the probability is $1/7!$.

3. (15 points) Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$.

(a) Compute the probability density function for $\min\{X_1, X_2, X_3\}$. **ANSWER:** This is exponential with rate $\lambda = 3$ so the density is $3e^{-3t}$ on $[0, \infty)$ (and zero elsewhere).

(b) Compute the expectation $E[\max\{X_1, X_2, X_3\}]$. **ANSWER:** This is the radioactive decay problem. Time first decay is exponential with rate 3, subsequent time until next decay is exponential with rate 2, and subsequent time until next decay is exponential with rate 1. Summing the expected times gives $\frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$.

4. (10 points) Suppose that X and Y are independent random variables, each of which has a probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$.

(a) Give the probability density function for $A = (X + Y)/2$. **ANSWER:** The average of independent Cauchy random variables is again Cauchy, so $f_A(x) = \frac{1}{\pi(1+x^2)}$

(b) Give the probability density function for $B = X - Y$. **ANSWER:** By symmetry, $-Y$ has same law as Y, so $X - Y$ has the same law as $2A = X + Y$. Tbus $f_B(x) = f_{2A}(x) = \frac{1}{2}f_B(x/2) = \frac{1}{2\pi\left(1+(x/2)^2\right)}$.

5. (15 points) Let C be fraction of students in a very large population that will say (when asked) that they prefer curry to pizza. Imagine that you start out knowing nothing about C, so that your Bayesian prior for C is uniform on $[0, 1]$. Then you select a student uniformly from the population and ask what that student prefers. You independently repeat this experiment two more times. You find that two students prefer curry and one prefers pizza.

(a) *Given* what you have learned from these three answers, give a revised probabilty density function f_C for the unknown quantity C (i.e., a Bayesian posterior). **ANSWER:** This is Beta with parameters $a = 2 + 1 = 3$ and $b = 1 + 1 = 2$. Compute $B(3, 2) = 2!1!/4! = 1/12$. So $x^2(1 - x)B(2, 3) = 12x^2(1 - x)$ on the interval $[0, 1]$.

(b) According to your Bayesian prior, the expected value of C was $1/2$. *Given* what you learned from the three answers, what is your *revised expectation* of the value C? **ANSWER:** The expectation of a Beta (a, b) random variable is $a/(a + b)$ which in this case is $3/5$.

6. (15 points) Let $X_1, X_2, \ldots, X_{100}$ be independent random variables, each of which is equal to 1 with probability 1/2 and 0 with probability 1/2. Write $S = \sum_{j=1}^{100} X_j$.

(a) Compute $E[S]$ and $\mathrm{Var}[S]$. **ANSWER:** S is binomial (n, p) and $E[S] = np = 50$ and $\mathrm{Var}[S] = npq = 25$.

(b) Use a normal random variable to approximate $P(S > 60)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** 60 is two standard deviations above the mean; by normal approximation $P(S > 60) \approx \int_2^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \Phi(2) = \Phi(-2)$.

7. (20 points) The residents of a certain planet are careful with nuclear weapons, but regional nuclear wars still occur. The times at which these wars occur form a Poisson point process with rate λ equal to one per thousand years. So the expected number of nuclear wars during any 1000 year period is 1.

(a) Compute the probability that there will be exactly three nuclear wars during the next 2000 years. **ANSWER:** This the probability a Poisson with parameter $\lambda = 2$ is equal to $k = 3$, which is $e^{-\lambda}\lambda^k/k! = e^{-2}2^3/3! = \frac{4}{3e^2}$.

(b) Let X be the number of millenia until the third nuclear war. (In other words, $1000X$ is the number of years until the third nuclear war.) Give the probability density function for f_X. **ANSWER:** This is Gamma with parameter $\lambda = 1$ and $n = 3$, which comes to $x^2e^{-x}/2$.

(c) Let Y be the number of nuclear wars that will occur during the next 5000 years. Compute the variance of Y. **ANSWER:** This is the variance of a Poisson random variable with parameter $\lambda = 5$, which is 5.

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