18.440 Midterm 2 Solutions, Spring 2011

1. (20 points) Jill polishes her resume and sends it to 900 companies she finds on monster.com. Each company responds with probability .1 (independently of what all the other companies do). Let R be the number of companies that respond.

(a) Compute the expectation of R (give an exact number).

ANSWER: $900 \cdot .01 = 90$

(b) Compute the standard deviation of R (given an exact number).

ANSWER: $\sqrt{900 \cdot .1 \cdot .9} = 9$

(c) Use a normal random variable approximation to estimate the probability $P\{R > 113\} = P\{R \geq 114\}$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer.

ANSWER: 114 is $\frac{114-90}{9} = \frac{24}{9} = 8/3$ standard deviations above the mean. So $P\{R \geq 114\} \approx 1 - \Phi(8/3)$. (Could also replace 114 or 113 or by 113.5. Would the latter give a better approximation?)

2. (20 points) Let X_1, X_2, and X_3 be independent uniform random variables on $[0, 1]$.

(a) Write $X = \max\{X_1, X_2, X_3\}$. Compute $P\{X \leq a\}$ for $a \in [0, 1]$.

ANSWER:
$P\{X \leq a\} = F_X(a) = P\{X_1 \leq a\}P\{X_2 \leq a\}P\{X_3 \leq a\} = a^3$ for $a \in [0, 1]$.

(b) Compute the probability density function for X on the interval $[0, 1]$.

ANSWER: $f_X(a) = F_X'(a) = 3a^2$ for $a \in [0, 1]$

(c) Compute the variance of the first variable X_1.

ANSWER: $\mathrm{Var}(X_1) = \frac{1}{12}$

(d) Compute the following covariance: $\mathrm{Cov}(X_1 + X_2, X_2 + X_3)$.

ANSWER: Using bilinearity of covariance, this is $\mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_2, X_2) + \mathrm{Cov}(X_2, X_3)$. All of these terms are zero except for $\mathrm{Cov}(X_2, X_2) = \frac{1}{12}$.

3. (10 points) Toss 3 fair coins independently.

(a) What is the *conditional* expected number of heads given that the first coin comes up heads?

ANSWER: Given first coin heads, each of second and third has .5 chance to be heads. Conditional expectation is 2.

(b) What is the *conditional* expected number of heads given that there are at least two heads among the three tosses.

ANSWER: A priori, have 3/8 chance to have 2 heads and 1/8 chance to have 3 heads. Conditioned on having either 2 or 3, there is a 3/4 chance to have 2 heads and a 1/4 chance to have three heads. So conditional expectation is $\frac{9}{4}$.

4. (10 points) Suppose that the amount of time until a certain radioactive particle decays is exponential with parameter λ. If there are three such particles, and their decay times are independent of each other, what is the expected amount of time until all three particles have decayed?

ANSWER: Time till first one decays is exponential with parameter 3λ. Subsequent time till next one decays is exponential with parameter 2λ. Subsequent time until last one decays is exponential with parameter λ. Expected sum of these three times is $\frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda}$.

5. (10 points) Let X be the number on a standard die roll (so X is chosen uniformly from the set $\{1, 2, 3, 4, 5, 6\}$).

(a) What is the moment generating function $M_X(t)$?

ANSWER: $M_X(t) = E[e^{Xt}] = \frac{1}{6}\big(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\big)$.

(b) Suppose that ten dice are rolled independently and Y is the sum of the numbers on all the dice. What is the moment generating function $M_Y(t)$?

ANSWER:
$M_Y(t) = (M_X(t))^{10} = \Big(\frac{1}{6}\big(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\big)\Big)^{10}$.

6. (20 points) On a certain hiking trail, it is well known that the lion, tiger, and bear attacks are independent Poisson point processes with respective λ values of .1/hour, .2/hour, and .3/hour. Let T be the number of hours until the first animal of any kind attacks.

(a) What is the probability that there are no lion attacks during the first hour?

ANSWER: $e^{-.1}$

(b) What is the probability density function for T?

ANSWER: Set of all attacks is a Poisson point process with $\lambda = .1 + .2 + .3 = .6$. So density is $f_T(t) = 0.6e^{-0.6t}$ for $t > 0$.

(c) What is the expected amount of time until the first tiger attack?

ANSWER: Expected amount of time until the first tiger attack is $1/.2 = 5$ hours.

(d) What is the distribution of the time until the fifth attack by any animal? (Give both the name of the distribution and an explicit formula.)

ANSWER: Sum of five independent exponentials of parameter .6 is a Gamma distribution with parameters $\alpha = 5$ and $\lambda = .6$. The density is $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ for $x > 0$.

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