18.600 Midterm 2, Fall 2019: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

NAME: ______________________________________

1. (20 points)

(a) Melissa is applying to 20 different out-of-state medical schools. Because of her excellent GPA/MCAT/essays, her chance of being accepted to each school is 1/20, and the decisions at the 20 schools are independent of each other. Using a Poisson approximation, estimate the probability that Melissa will be accepted to at least two of these schools.

(b) Jill is applying to 25 different out-of-state medical schools and has a 1/5 chance (independently) of being invited for an interview at each school. Let X be the number of medical schools at which she is invited to interview. Compute $E[X]$ and $\mathrm{Var}[X]$.

(c) Using a normal approximation, roughly approximate the probability that Jill is invited to interview at fewer than 2.5 schools. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer.

2. (20 points) A room has four lightbulbs, each of which will burn out at a random time. Let X_1, X_2, X_3, X_4 be the burnout times, and assume they are independent exponential random variables with parameter $\lambda = 1$. Write

1. $X = X_1 + X_2 + X_3 + X_4$.
2. $Y = \min\{X_1, X_2, X_3, X_4\}$, i.e., Y is time when first bulb burns out.
3. $Z = \max\{X_1, X_2, X_3, X_4\}$, i.e., Z is time when last bulb burns out.

Compute the following:

(a) The probablity density function f_X.

(b) The probability density function f_Y.

(c) The expectation $E[Z]$.

(d) The covariance $\mathrm{Cov}(Y, Z)$. (Hint: use memoryless property.)

3. (20 points) Five applicants are applying for a job, and an interviewer gives each applicant a score between 0 and 1. Call these scores $X_1, X_2, \ldots, X_5$ and assume that they are i.i.d. uniform random variables on $[0,1]$. The top applicant has score $Y = \max\{X_1, X_2, \ldots, X_5\}$, and the second to the top has score Z, which we define to be the *second* largest of the X_i. Compute the following:

(a) The cumulative distribution function $F_Y(r)$ for $r \in [0,1]$.

(b) The density function f_Y.

(c) The density function f_Z and the value $E[Z]$. **NOTE:** If you remember what this means, you may use the fact that a Beta (a,b) random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a,b)$, where $B(a,b) = (a-1)!(b-1)!/(a+b-1)!$.

(d) The probability $P(X_2 > 2X_1)$ (i.e., probability second candidate's score is more than than double first candidate's score).

4. (15 points) Let X and Y be independent random variables with density function given by $\frac{1}{\pi(1+x^2)}$.

(a) Compute $P(X < 1)$.

(b) Compute the probability density function for the random variable $Z = (X - Y)/2$.

(c) Compute $E[e^{-X^2-Y^2}]$. You can leave your answer as a double integral—no need to evaluate it explicitly.

5. (10 points) Let $X_1, X_2, X_3, \ldots, X_{10}$ be the outcomes of independent standard die rolls—so each takes one of the values in $\{1, 2, 3, 4, 5, 6\}$, each with equal probability. Write $S = X_1 + X_2 + \ldots + X_{10}$. Compute the following:

(a) The moment generating function $M_{X_1}(t)$.

(b) The moment generating function $M_S(t)$.

6. (15 points) Let X and Y be be random variables with joint density function $f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-x^2-y^2}$. Write $Z = X + Y$.

(a) Compute $E[XY]$.

(b) Compute the conditional expectation $E[Y|Z]$. That is, express the random variable $E[Y|Z]$ in terms of Z.

(c) Compute the probability $P(X^2 + Y^2 \leq 4)$.