Math 704 10/6/04.

Goal: Descent Theorem.

Height $x \in \mathbb{Q}$ $x = \frac{m}{n}$ in lowest terms.

(big H) $H(x) = \max(|m|, |n|)$.

$x = 1$ $H(x) = 1$ $x = \frac{99999}{100{,}000}$ $H(x) = 100{,}000$

Finiteness Property of the Height

The set of all $x \in \mathbb{Q}$ s.t. $H(x) \leq K$ is a finite set.

Proof. If $H(x) \leq K \rightarrow |m| \leq K$, $|n| \leq K$
so there are finite ways to choose x.

Height of Points.

$P = (x, y)$ then height of $P = H(x)$.

Logarithmic Height. (little h).

$h(x) = \log H(x)$.

Finiteness Property of Rational Points.

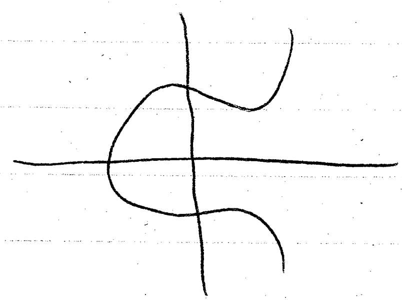
For any positive number M

$\{ P \in C(\mathbb{Q}) : H(P) \leq M \}$ is a finite set.

--- $\{$ " $h(P)$ " "

Finitely many ways to choose the x-coordinate

Two possible y-coords.



<u>Point at Infinity</u>

$H(\mathcal{O}) = 1 \qquad h(\mathcal{O}) = 0.$

<u>Lemma 1</u>: For ~~add~~ any positive number M

$\{ P \in C(\mathbb{Q}) : h(P) \le M \}$ is finite.

<u>Lemma 2</u>: Let P_0 be a fixed rational point on C.

There is a constant depending on P_0 and a, b, c

s.t.

$\forall P \in C(\mathbb{Q}) \quad h(P + P_0) \le 2h(P) + K_0.$

<u>Lemma 3</u>: There is a constant K depending on a, b, c

s.t. $\forall P \in C(\mathbb{Q})$

$h(2P) \ge 4h(P) - K.$

<u>Lemma 4</u>: The index $(C(\mathbb{Q}), 2C(\mathbb{Q}))$ is finite.

<u>Multiplication-by-m map</u>

For any commutative group Γ,

$\Gamma \to \Gamma \qquad P \longrightarrow \underbrace{P + P + \cdots + P}_{m \text{ terms}} = mP$

is a homomorphism, and the image is a subgroup $m\Gamma$ of Γ.

Descent Theorem

Let Γ be a commutative group.
Suppose that there is a function
$h: \Gamma \longrightarrow [0, \infty)$ with the following properties:

a) For every real number M the set $\{P \in \Gamma \mid h(P) \le M\}$ is finite.

b). For every $P_0 \in \Gamma$, there is a constant k_0 s.t.
$h(P + P_0) \le 2h(P) + k_0 \quad \forall P \in \Gamma$.

c) There is a constant k so that $h(2P) \ge 4h(P) - k$. $\forall P \in \Gamma$

d) The subgroup 2Γ has finite index in Γ.

Then Γ is finitely generated.

---

1 Take a representative for each coset of 2Γ in Γ.
There are finitely many cosets, say n, so let $Q_1, Q_2, \ldots, Q_n$ be the representatives.
For any $P \in \Gamma$, there is an index i_1 depending on P s.t. $P - Q_{i_1} \in 2\Gamma$.

$P - Q_{i_1} = 2P_1$ for some $P_1 \in \Gamma$.

$P_1 - Q_{i_2} = 2P_2$

$P_2 - Q_{i_3} = 2P_3$ where Q_i's are chosen from $Q_1, \ldots, Q_n$, and

$\vdots$

$P_{(m-1)} - Q_{i_m} = 2P_m$ $\quad P_1, \ldots, P_m \in \Gamma$.

② $P = Q_{i_1} + 2P_1$

$P_1 = Q_{i_2} + 2P_2$

$$P = Q_{i_1} + 2Q_{i_2} + 4Q_{i_3} + \cdots + 2^{m-1}Q_{i_m} + 2^m P_m$$

P is in the subgroup of Γ generated by ~~Q~~ the Q_i's and P_m.

③ Take one of the P_j's in the sequence of $P, P_1, P_2, \ldots$ and examine the relation between $h(P_j)$ and $h(P_{j-1})$.

$$h(P - Q_i) \le 2h(P) + k_i \quad \forall P \in \Gamma.$$

Do this for all Q_i, $1 \le i \le n$.

Let k' be the largest of the k_i's.

$$h(P - Q_i) \le 2h(P) + k' \quad \forall P \in \Gamma,\ 1 \le i \le n.$$

④ Let K be the constant from (c).

$$4h(P_j) \le h(2P_j) + K. = h(P_{j-1} - Q_{i_j}) + K.$$

$$\le 2h(P_{j-1}) + k' + K.$$

$$h(P_j) \le \tfrac{1}{2}h(P_{j-1}) + \frac{k'+k}{4}.$$

$$= \tfrac{3}{4}h(P_{j-1}) - \tfrac{1}{4}\left(h(P_{j-1}) - k'+k\right)$$

<u>Bottom Line</u> If $h(P_{j-1}) \ge k' + K$ then $h(P_j) \le \frac{3}{4}h(P_{j-1})$.

(5). In the sequence of points $P, P_1, P_2, \ldots$
Each point has a height smaller than the previous pt.
(if we haven't yet reached P_m)
Eventually we reach a point P_m

$$h(P_m) \le K' + K.$$

<u>Conclusion.</u> We have shown that for all elements $P \in \Gamma$, P can be written as

$$P = a_1 Q_1 + a_2 Q_2 + \cdots + a_n Q_n + 2^m R. \quad R \in \Gamma$$

satisfying

$$h(R) \le K + K'.$$

* Hence the set

$$\{Q_1, \ldots, Q_m\} \cup \{R \in \Gamma : h(R) \le K + K'\}$$

will generate Γ.

Therefore Γ is finitely generated.