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SINGULAR CUBICS.

$C: y^2 = x^3 + ax^2 + bx + c$, $= f(x)$ has a multiple root.

$C_{ns} = \{ P \in C \mid P$ is not a point of singularity $\}$.

Claim: $C_{ns}(\mathbb{Q})$ forms a group.

$P + Q = O_{ns} * (P * Q)$ $\quad P, Q, O_{ns} \in C_{ns}(\mathbb{Q})$

Need to show closure.

<u>Lemma</u>: A line $\ell: \lambda x + \nu$ ~~through the~~ that intersects C in a singular point $P = (x_0, y_0)$ intersects C twice at P.

$$F(x,y) = y^2 - f(x) = 0$$
$$\left.\frac{\partial F}{\partial x}\right|_{x=x_0} = -f'(x_0) \quad \left.\frac{\partial F}{\partial y}\right|_{y=y_0} = 2y_0$$

If $P = (x_0, y_0)$ then $y_0 = 0$ and x_0 is a double root of f.

$(x_0, 0) \in \ell$

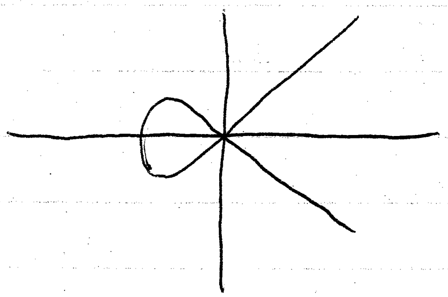
$y^2 = f(x) \iff f(x) - y^2 = 0$

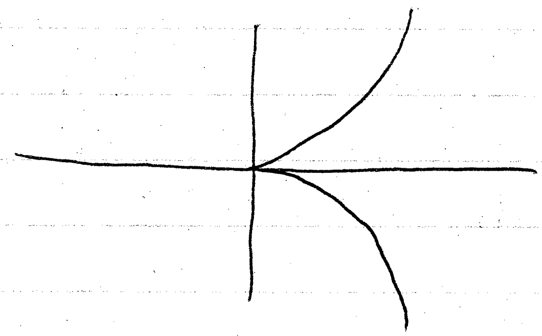
at ~~on~~ $(\lambda x + \nu)^2 = 0$ $\quad$ So ℓ intersects C at P at least twice.

Corollary. There is at most 1 singular point of C.

Consider:

$C_1 : y^2 = x^3 + x^2$ $\qquad$ $C_2 : y^2 = x^3$





Theorem:

(i) $C_{1ns}(\mathbb{Q}) \cong \mathbb{Q}^*$

(ii) $C_{2ns}(\mathbb{Q}) \cong \mathbb{Q}^+$

Proof of (i): Define a map $\phi : C_{1ns}(\mathbb{Q}) \longrightarrow \mathbb{Q}^*$

$$\phi(P) = \begin{cases} \frac{y-x}{y+x} & \text{if } P = (x,y) \\ 1 & \text{if } P = \mathcal{O} \end{cases}$$

Claim: ϕ is an isomorphism.

Want ϕ bijective

$t = \frac{y-x}{y+x}$ then $y = \left(\frac{1+t}{1-t}\right)x$

Substituting: $x^3 + \left(1-\left(\frac{1+t}{1-t}\right)^2\right) x^2 = 0$

$$x = \frac{(1+t)^2 - (1-t)^2}{(1-t)^2} = \frac{4t}{(1-t)^2}$$

$$\psi : \mathbb{Q}^\times \longrightarrow C_{ns}(\mathbb{Q})$$

$$\psi(t) = \begin{cases} \left(\frac{4t}{(1-t)^2}, \frac{4t(1+t)}{(1-t)^3}\right) & \text{if } t \neq 1 \\ \mathcal{O} & t = 1 \end{cases}$$

$\phi(\psi(t)) = t \qquad \psi(\phi(P)) = P$

$\Rightarrow \phi, \psi$ are bijective.

ϕ is an isomorphism $\iff$ ψ is an isomorphism

$$\psi\left(\frac{1}{t}\right) = \left(\frac{4t^{-1}}{(1-t^{-1})^2}, \frac{4t^{-1}(1+t^{-1})}{(1-t^{-1})^3}\right)$$

$$= \left(\frac{4t}{(t-1)^2}, \frac{-4t(1+t)}{(1-t)^3}\right) = -\psi(t).$$

Given $P_1, P_2, P_3 \in C_{ns}(\mathbb{Q})$ we know

$P_1 + P_2 + P_3 = \mathcal{O} \iff$ They are collinear

Say $P_i = (x_i, y_i)$

ℓ containing P_1, P_2 $(P_1 \neq P_2)$ $\quad (x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$

*

$P_1P_2P_3$ collinear if $x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$ (*)

Want to prove: if $t_1t_2t_3 = 1$ $\quad t_1, t_2, t_3 \in \mathbb{Q}^*$
then
$\psi(t_1) + \psi(t_2) + \psi(t_3) = \mathcal{O}$.

$\Rightarrow \forall\, t_1, t_2 \in \mathbb{Q}^*$

$$\psi(t_1t_2) + \psi((t_1t_2)^{-1}) = \psi(t_1t_2) + \psi(t_1t_2)^{-1}$$

$$= 1 = \psi(t_1) + \psi(t_2) + \psi((t_1t_2)^{-1})$$

$$\Rightarrow \psi(t_1t_2) = \psi(t_1) + \psi(t_2)$$

$$\psi(t) = \left(\frac{4t}{(1-t)^2}, \frac{4t(1+t)}{(1-t)^3} \right) \qquad t \neq 1$$

Substitute into (*) with $P_i = \psi(t_i)$

$$\text{LHS}(*) = \frac{32(t_1-t_2)(t_1-t_3)(t_2-t_3)(t_1t_2t_3-1)}{(1-t_1)^3(1-t_2)^3(1+t_3)^3}$$

$= \mathcal{O}$ if $t_1t_2t_3 = 1$, $t_1, t_2, t_3 \neq 1$. and t_i distinct.

This prove $t_1t_2t_3 = 1$ $\Rightarrow \psi(t_1) + \psi(t_2) + \psi(t_3) = \mathcal{O}$
$t_1, t_2, t_3 \neq 1$
and t_i distinct.

$\Rightarrow \Psi$ is an isomorphism.
so Φ is an isomorphism.

$\mathbb{Q}^*$ is not finitely generated.

$$\left\{ \frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k} \right\} \qquad \prod_{i=1}^{k} \left(\frac{m_i}{n_i}\right)^{\alpha_i}$$