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1 Fiber Products

We are trying to construct the fiber product, namely, given maps f, g, find the universal object W such that

$$\begin{array}{ccc} X & \leftarrow & W \\ f\downarrow & & \downarrow \\ Y & \overset{g}{\leftarrow} & Z \end{array}$$

Let T, R, S be the coordinate rings for X, Y, Z respectively. Then the right thing to put for the ring of W would be $T \otimes_R S$. However, note that $T \otimes_R S$ is not necessarily an integral domain, or in other words, W may not be irreducible.

2 Back to it

Thm. Let $f : X \to Y$ be a dominant morphism. Then there exists a nonempty $U \subset Y$ open such that

1. $U \subset f(X)$
2. $\forall W \subset Y$ irred. closed, $W \cap U \neq \emptyset$, and irreducible component Z of $f^{-1}(W)$ with nonempty intersection with $f^{-1}(U)$, then $\dim Z = \dim W + r$ where $r = \dim X - \dim Y$.

Pf. Special case: suppose there exists a factorization $X \xrightarrow{\pi} Y \times \mathbb{A}^r \xrightarrow{p_1} Y$ where π is finite and surjective. In this case, we can pick $U = Y$. Clearly (1) is okay. As for (2), let $Z \subset f^{-1}(W) \subset X$ be an irreducible component, $\dim Z \geq \dim W + r$. Now let $\overline{Z} \subset Y \times \mathbb{A}^r$ be $\pi(Z)$. We know that $\overline{Z}$ is still closed and irreducible. Also, $\dim \overline{Z} = \dim Z$ since the map $k(\overline{Z}) \to k(Z)$ is finite, which means this is a finite, meaning algebraic, field extension, so they have the same transcendence degree.

Now we have $W \times \mathbb{A}^r$ which actually is (in this case) the fiber product $W \times_Y (Y \times \mathbb{A}^r)$, so we get a map $\overline{Z} \hookrightarrow W \times \mathbb{A}^r$, so $\dim \overline{Z} \leq \dim W + r$ which proves $\dim Z = \dim W + r$.

Next case: suppose X, Y affine. We have $R = \Gamma(X, O_X), S = \Gamma(Y, O_Y)$, $S \xrightarrow{f^*} R$. We want a factorization of this by $S \to S[X_1, \ldots, X_r] \to R$ where the second map is finite and *injective* (this corresponds to a finite, surjective morphism).

Claim: After replacing S by S_g and R by $R_{f^*(g)}$ for some nonzero $g \in S$ we can find such a factorization. Let $K = \text{Frac }(S), R^* = R \otimes_S K$. Then we get a factorization of the map $K \to R^*$ by $K \to K[X_1, \ldots, X_r] \to R^*$. Now Frac $(R^*) =$ Frac (R) and so we get $\text{tr.deg}_k(\text{Frac }(R^*)) = \dim X$, and $\text{tr.deg}_k(K) = \dim(Y)$, so $\text{tr.deg}_K(\text{Frac }(R^*)) = r$. By Noether normalization, we can find a map $K[X_1, \ldots, X_r] \xrightarrow{\phi} R^*$ which is integral, and thus we get the nice factorization we want. We want to bring this back to a factorization

$StoS[X_1, \ldots, X_r] \to R$. Now for each i, $\phi(X_i) \in R^*$ lies in $S_{g_i} \otimes_S R$ for some $g_i \in S$ (this is because $K = \varinjlim_{g \in S} S_g$ and limits and tensors commute. This lifting of ϕ (call it $\tilde{\phi}$ may not be integral (which would be nice, as it would finish things off).

However, further localization can make $\tilde{\phi}$ integral; we basically localize to allow the denominators we need. The reason we can do this only finitely many times is that R is *finitely generated.*

Now the general case. Suppose we have $X \xrightarrow{f} Y$, and we can assume Y is affine. Choose an affine cover $X = \cup_i X_i$. Then we get $U_i \subset Y$ such that the theorem holds for $f : X_i \to Y$. Then, let $U = \cap_i U_i$ and this works for X.

Def. A morphism $f : X \to Y$ is *birational* if it is dominant and $k(Y) \to k(X)$ is an isomorphism.

Ex. The blow-up map is birational, since the blow-up $X \to \mathbb{A}^n$ is the same everywhere but at the origin, but $k(X)$ only depends on a dense open set.

Ex. $\mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ where $t \mapsto (t^2, t^3)$.

Thm. If $f : X \to Y$ is birational, then there exists a nonempty open $U \subset Y$ such that $f^{-1}(U) \xrightarrow{f} U$ is an $\cong$.

Pf. We can assume Y is affine. If X is affne we're done. If $U \subset X$ is an open affine with coordinate ring R, let $W = \overline{f(X - U)}$. The components of W have smaller dimension than Y, so in particular $W \neq Y$. Choose $g \in \Gamma(Y, O_Y)$ such that $D(g) \subset W^C$. etc.

3 Complete Varieties

or "Proper Varieties." One motivating thing here is to say that $\mathbb{C}^n$ should not be compact, but $\mathbb{A}^n$ *is* compact under regular definitions, so we need a different kind of definition to give us what we want.

If we're working in the category of Hausdorff topological spaces, then X is compact if and only if for every space Y, the map $X \times Y \xrightarrow{p_2} Y$ is a closed map. So this is the way we're going to try to define compactness for varieties.

Def. Let X be a variety. Then X is *complete* if for every variety Y, the map $X \times Y \xrightarrow{p_2} Y$ is closed.

Ex. $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1$ is not closed because we can start with $V(xy - 1)$ and end up with $\mathbb{A}^1 - \{0\}$ so $\mathbb{A}^1$ is *not* complete. Good! This is what we wanted.