5. Parabolic induction and restriction functors for rational Cherednik algebras

5.1. **A geometric approach to rational Cherednik algebras.** An important property of the rational Cherednik algebra $H_{1,c}(G,\mathfrak{h})$ is that it can be sheafified, as an algebra, over $\mathfrak{h}/G$ (see [E1]). More specifically, the usual sheafification of $H_{1,c}(G,\mathfrak{h})$ as a $\mathcal{O}_{\mathfrak{h}/G}$-module is in fact a quasicoherent sheaf of algebras, $H_{1,c,G,\mathfrak{h}}$. Namely, for every affine open subset $U\subset \mathfrak{h}/G$, the algebra of sections $H_{1,c,G,\mathfrak{h}}(U)$ is, by definition, $\mathbb{C}[U]\otimes_{\mathbb{C}[\mathfrak{h}]^G} H_{1,c}(G,\mathfrak{h})$.

The same sheaf can be defined more geometrically as follows (see [E1], Section 2.9). Let $\widetilde{U}$ be the preimage of U in $\mathfrak{h}$. Then the algebra $H_{1,c,G,\mathfrak{h}}(U)$ is the algebra of linear operators on $\mathcal{O}(\widetilde{U})$ generated by $\mathcal{O}(\widetilde{U})$, the group G, and Dunkl operators

$$\partial_a-\sum_{s\in\mathcal{S}}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{\alpha_s}(1-s),\ \text{where } a\in\mathfrak{h}.$$

5.2. **Completion of rational Cherednik algebras.** For any $b\in\mathfrak{h}$ we can define the completion $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ to be the algebra of sections of the sheaf $H_{1,c,G,\mathfrak{h}}$ on the formal neighborhood of the image of b in $\mathfrak{h}/G$. Namely, $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ is generated by regular functions on the formal neighborhood of the G-orbit of b, the group G, and Dunkl operators.

The algebra $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ inherits from $H_{1,c}(G,\mathfrak{h})$ the natural filtration $F^\bullet$ by order of differential operators, and each of the spaces $F^n\widehat{H_{1,c}}(G,\mathfrak{h})_b$ has a projective limit topology; the whole algebra is then equipped with the topology of the nested union (or inductive limit).

Consider the completion of the rational Cherednik algebra at zero, $\widehat{H_{1,c}}(G,\mathfrak{h})_0$. It naturally contains the algebra $\mathbb{C}[[\mathfrak{h}]]$. Define the category $\widehat{\mathcal{O}}_c(G,\mathfrak{h})$ of representations of $\widehat{H_{1,c}}(G,\mathfrak{h})_0$ which are finitely generated over $\mathbb{C}[[\mathfrak{h}]]_0=\mathbb{C}[[\mathfrak{h}]]$.

We have a completion functor $\widehat{\ }$: $\mathcal{O}_c(G,\mathfrak{h})\to\widehat{\mathcal{O}}_c(G,\mathfrak{h})$, defined by

$$\widehat{M}=\widehat{H_{1,c}}(G,\mathfrak{h})_0\otimes_{H_{1,c}(G,\mathfrak{h})}M=\mathbb{C}[[\mathfrak{h}]]\otimes_{\mathbb{C}[\mathfrak{h}]}M.$$

Also, for $N\in\widehat{\mathcal{O}}_c(G,\mathfrak{h})$, let $E(N)$ be the subspace spanned by generalized eigenvectors of $\mathbf{h}$ in N where $\mathbf{h}$ is defined by (3.2). Then it is easy to see that $E(N)\in\mathcal{O}_c(G,\mathfrak{h})_0$.

Theorem 5.1. *The restriction of the completion functor* $\widehat{\ }$ *to* $\mathcal{O}_c(G,\mathfrak{h})_0$ *is an equivalence of categories* $\mathcal{O}_c(G,\mathfrak{h})_0\to\widehat{\mathcal{O}}_c(G,\mathfrak{h})$. *The inverse equivalence is given by the functor* E.

Proof. It is clear that $M\subset\widehat{M}$, so $M\subset E(\widehat{M})$ (as M is spanned by generalized eigenvectors of $\mathbf{h}$). Let us demonstrate the opposite inclusion. Pick generators $m_1,\dots,m_r$ of M which are generalized eigenvectors of $\mathbf{h}$ with eigenvalues $\mu_1,\dots,\mu_r$. Let $0\neq v\in E(\widehat{M})$. Then $v=\sum_i f_i m_i$, where $f_i\in\mathbb{C}[[\mathfrak{h}]]$. Assume that $(\mathbf{h}-\mu)^N v=0$ for some N. Then $v=\sum_i f_i^{(\mu-\mu_i)}m_i$, where for $f\in\mathbb{C}[[\mathfrak{h}]]$ we denote by $f^{(d)}$ the degree d part of f. Thus $v\in M$, so $M=E(\widehat{M})$.

It remains to show that $\widehat{E(N)}=N$, i.e. that N is the closure of $E(N)$. In other words, letting $\mathfrak{m}$ denote the maximal ideal in $\mathbb{C}[[\mathfrak{h}]]$, we need to show that the natural map $E(N)\to N/\mathfrak{m}^jN$ is surjective for every j.

To do so, note that $\mathbf{h}$ preserves the descending filtration of N by subspaces $\mathfrak{m}^jN$. On the other hand, the successive quotients of these subspaces, $\mathfrak{m}^jN/\mathfrak{m}^{j+1}N$, are finite dimensional, which implies that $\mathbf{h}$ acts locally finitely on their direct sum $\mathrm{gr}N$, and moreover each generalized eigenspace is finite dimensional. Now for each $\beta \in \mathbb{C}$ denote by $N_{j,\beta}$ the generalized β-eigenspace of $\mathbf{h}$ in $N/\mathfrak{m}^j N$. We have surjective homomorphisms $N_{j+1,\beta} \to N_{j,\beta}$, and for large enough j they are isomorphisms. This implies that the map $E(N) \to N/\mathfrak{m}^j N$ is surjective for every j, as desired. □

Example. Suppose that $c = 0$. Then Theorem 5.1 specializes to the well known fact that the category of G-equivariant local systems on $\mathfrak{h}$ with a locally nilpotent action of partial differentiations is equivalent to the category of all G-equivariant local systems on the formal neighborhood of zero in $\mathfrak{h}$. In fact, both categories in this case are equivalent to the category of finite dimensional representations of G.

We can now define the composition functor $\mathcal{J} : \mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_0$, by the formula $\mathcal{J}(M) = E(\widehat{M})$. The functor $\mathcal{J}$ is called the Jacquet functor ([Gi2]).

5.3. **The duality functor.** Recall that in Section 3.11, for any $H_{1,c}(G, \mathfrak{h})$-module M, the full dual space M^* is naturally an $H_{1,\bar{c}}(G, \mathfrak{h}^*)$-module, via $\pi_{M^*}(a) = \pi_M(\gamma(a))^*$.

It is clear that the duality functor $*$ defines an equivalence between the category $\mathcal{O}_c(G, \mathfrak{h})_0$ and $\widehat{\mathcal{O}}_{\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}}$, and that $M^\dagger = E(M^*)$ (where $M^\dagger$ is the contragredient, or restricted dual module to M defined in Section 3.11).

5.4. **Generalized Jacquet functors.**

Proposition 5.2. *For any $M \in \widehat{\mathcal{O}}_c(G, \mathfrak{h})$, a vector $v \in M$ is $\mathbf{h}$-finite if and only if it is $\mathfrak{h}$-nilpotent.*

Proof. The "if" part follows from Theorem 3.20. To prove the "only if" part, assume that $(\mathbf{h} - \mu)^N v = 0$. Then for any $u \in S^r\mathfrak{h} \cdot v$, we have $(\mathbf{h} - \mu + r)^N u = 0$. But by Theorem 5.1, the real parts of generalized eigenvalues of $\mathbf{h}$ in M are bounded below. Hence $S^r\mathfrak{h} \cdot v = 0$ for large enough r, as desired. □

According to Proposition 5.2, the functor E can be alternatively defined by setting $E(M)$ to be the subspace of M which is locally nilpotent under the action of $\mathfrak{h}$.

This gives rise to the following generalization of E: for any $\lambda \in \mathfrak{h}^*$ we define the functor $E_\lambda : \widehat{\mathcal{O}}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_\lambda$ by setting $E_\lambda(M)$ to be the space of generalized eigenvectors of $\mathbb{C}[\mathfrak{h}^*]^G$ in M with eigenvalue λ. This way, we have $E_0 = E$.

We can also define the generalized Jacquet functor $\mathcal{J}_\lambda : \mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_\lambda$ by the formula $\mathcal{J}_\lambda(M) = E_\lambda(\widehat{M})$. Then we have $\mathcal{J}_0 = \mathcal{J}$, and one can show that the restriction of $\mathcal{J}_\lambda$ to $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the identity functor.

5.5. **The centralizer construction.** For a finite group H, let $\mathsf{e}_H = |H|^{-1} \sum_{g \in H} g$ be the symmetrizer of H.

If $G \supset H$ are finite groups, and A is an algebra containing $\mathbb{C}[H]$, then define the algebra $Z(G, H, A)$ to be the centralizer $\mathrm{End}_A(P)$ of A in the right A-module $P = \mathrm{F}un_H(G, A)$ of H-invariant A-valued functions on G, i.e. such functions $f : G \to A$ that $f(hg) = hf(g)$. Clearly, P is a free A-module of rank $|G/H|$, so the algebra $Z(G, H, A)$ is isomorphic to $\mathrm{M}at_{|G/H|}(A)$, but this isomorphism is not canonical.

The following lemma is trivial.

Lemma 5.3. *The functor $N \mapsto I(N) := P \otimes_A N = \mathrm{F}un_H(G, N)$ defines an equivalence of categories $A - \mathrm{mod} \to Z(G, H, A) - \mathrm{mod}$.*

5.6. **Completion of rational Cherednik algebras at arbitrary points of $\mathfrak{h}/G$.** The following result is, in essence, a consequence of the geometric approach to rational Cherednik algebras, described in Subsection 5.1. It should be regarded as a direct generalization to the case of Cherednik algebras of Theorem 8.6 of [L] for affine Hecke algebras.

Let $b \in \mathfrak{h}$. Abusing notation, denote the restriction of c to the set $\mathcal{S}_b$ of reflections in G_b also by c.

Theorem 5.4. *One has a natural isomorphism*

$$\theta : \widehat{H_{1,c}}(G, \mathfrak{h})_b \to Z(G, G_b, \widehat{H_{1,c}}(G_b, \mathfrak{h})_0),$$

defined by the following formulas. Suppose that $f \in P = \mathrm{Fun}_{G_b}(G, \widehat{H_{1,c}}(G_b, \mathfrak{h})_0)$. *Then*

$$(\theta(u)f)(w) = f(wu), u \in G;$$

for any $\alpha \in \mathfrak{h}^*$,

$$(\theta(x_\alpha)f)(w) = (x^{(b)}_{w\alpha} + (w\alpha, b))f(w),$$

where $x_\alpha \in \mathfrak{h}^* \subset H_{1,c}(G, \mathfrak{h})$, $x^{(b)}_\alpha \in \mathfrak{h}^* \subset H_{1,c}(G_b, \mathfrak{h})$ *are the elements corresponding to* α*; and for any* $a \in \mathfrak{h}$,

$$(\theta(y_a)f)(w) = y^{(b)}_{wa} f(w) - \sum_{s \in \mathcal{S}: s \notin G_b} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s} + \alpha_s(b)} (f(w) - f(sw)). \tag{5.1}$$

where $y_a \in \mathfrak{h} \subset H_{1,c}(G, \mathfrak{h})$, $y^{(b)}_a \in \mathfrak{h} \subset H_{1,c}(G_b, \mathfrak{h})$.

Proof. The proof is by a direct computation. We note that in the last formula, the fraction $\alpha_s(wa)/(x^{(b)}_{\alpha_s} + \alpha_s(b))$ is viewed as a power series (i.e., an element of $\mathbb{C}[[\mathfrak{h}]]$), and that only the entire sum, and not each summand separately, is in the centralizer algebra. $\square$

Remark. Let us explain how to see the existence of θ without writing explicit formulas, and how to guess the formula (5.1) for θ. It is explained in [E1] (see e.g. [E1], Section 2.9) that the sheaf of algebras obtained by sheafification of $H_{1,c}(G, \mathfrak{h})$ over $\mathfrak{h}/G$ is generated (on every affine open set in $\mathfrak{h}/G$) by regular functions on $\mathfrak{h}$, elements of G, and Dunkl operators. Therefore, this statement holds for formal neighborhoods, i.e., it is true on the formal neighborhood of the image in $\mathfrak{h}/G$ of any point $b \in \mathfrak{h}$. However, looking at the formula for Dunkl operators near b, we see that the summands corresponding to $s \in \mathcal{S}, s \notin G_b$ are actually regular at b, so they can be safely deleted without changing the generated algebra (as all regular functions on the formal neighborhood of b are included into the system of generators). But after these terms are deleted, what remains is nothing but the Dunkl operators for $(G_b, \mathfrak{h})$, which, together with functions on the formal neighborhood of b and the group G_b, generate the completion of $H_{1,c}(G_b, \mathfrak{h})$. This gives a construction of θ without using explicit formulas.

Also, this argument explains why θ should be defined by formula (5.1) of Theorem 5.4. Indeed, what this formula does is just restores the terms with $s \notin G_b$ that have been previously deleted.

The map θ defines an equivalence of categories

$$\theta_* : \widehat{H_{1,c}}(G, \mathfrak{h})_b - \mathrm{mod} \to Z(G, G_b, \widehat{H_{1,c}}(G_b, \mathfrak{h})_0) - \mathrm{mod}.$$

Corollary 5.5. *We have a natural equivalence of categories*

$$\psi_\lambda : \mathcal{O}_c(G, \mathfrak{h})_\lambda \to \mathcal{O}_c(G_\lambda, \mathfrak{h}/\mathfrak{h}^{G_\lambda})_0.$$

Proof. The category $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the category of modules over $H_{1,c}(G, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_{1,c}(G, \mathfrak{h})$ at λ. So it follows from Theorem 5.4 that we have an equivalence $\mathcal{O}_c(G, \mathfrak{h})_\lambda \to \mathcal{O}_c(G_\lambda, \mathfrak{h})_0$. Composing this equivalence with the equivalence $\zeta : \mathcal{O}_c(G_\lambda, \mathfrak{h})_0 \to \mathcal{O}_c(G_\lambda, \mathfrak{h}/\mathfrak{h}^{G_\lambda})_0$, we obtain the desired equivalence ψ_λ. □

Remark 5.6. Note that in this proof, we take the completion of $H_{1,c}(G, \mathfrak{h})$ at a point of $\lambda \in \mathfrak{h}^*$ rather than $b \in \mathfrak{h}$.

5.7. **The completion functor.** Let $\widehat{\mathcal{O}}_c(G, \mathfrak{h})^b$ be the category of modules over $\widehat{H_{1,c}}(G, \mathfrak{h})_b$ which are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_b$.

Proposition 5.7. *The duality functor $*$ defines an anti-equivalence of categories $\mathcal{O}_c(G, \mathfrak{h})_\lambda \to \widehat{\mathcal{O}}_{\bar{c}}(G, \mathfrak{h}^*)^\lambda$.*

Proof. This follows from the fact (already mentioned above) that $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the category of modules over $H_{1,c}(G, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_{1,c}(G, \mathfrak{h})$ at λ. □

Let us denote the functor inverse to $*$ also by $*$; it is the functor of continuous dual (in the formal series topology).

We have an exact functor of completion at b, $\mathcal{O}_c(G, \mathfrak{h})_0 \to \widehat{\mathcal{O}}_c(G, \mathfrak{h})^b$, $M \mapsto \widehat{M}_b$. We also have a functor $E^b : \widehat{\mathcal{O}}_c(G, \mathfrak{h})^b \to \mathcal{O}_c(G, \mathfrak{h})_0$ in the opposite direction, sending a module N to the space $E^b(N)$ of $\mathfrak{h}$-nilpotent vectors in N.

Proposition 5.8. *The functor E^b is right adjoint to the completion functor $\widehat{\ }_b$.*

Proof. We have

$$\mathrm{Hom}_{\widehat{H_{1,c}}(G,\mathfrak{h})_b}(\widehat{M}_b, N) = \mathrm{Hom}_{\widehat{H_{1,c}}(G,\mathfrak{h})_b}(\widehat{H_{1,c}}(G, \mathfrak{h})_b \otimes_{H_{1,c}(G,\mathfrak{h})} M, N)$$

$$= \mathrm{Hom}_{H_{1,c}(G,\mathfrak{h})}(M, N|_{H_{1,c}(G,\mathfrak{h})}) = \mathrm{Hom}_{H_{1,c}(G,\mathfrak{h})}(M, E^b(N)).$$

□

Remark 5.9. Recall that by Theorem 5.1, if $b = 0$ then these functors are not only adjoint but also inverse to each other.

Proposition 5.10. (i) *For $M \in \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_b$, one has $E^b(M^*) = (\widehat{M})^*$ in $\mathcal{O}_c(G, \mathfrak{h})_0$.*
(ii) *For $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, $(\widehat{M}_b)^* = E_b(M^*)$ in $\mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_b$.*
(iii) *The functors E_b, E^b are exact.*

Proof. (i),(ii) are straightforward from the definitions. (iii) follows from (i),(ii), since the completion functors are exact. □

5.8. **Parabolic induction and restriction functors for rational Cherednik algebras.** Theorem 5.4 allows us to define analogs of parabolic restriction functors for rational Cherednik algebras.

Namely, let $b \in \mathfrak{h}$, and $G_b = G'$. Define a functor $\mathrm{Res}_b : \mathcal{O}_c(G, \mathfrak{h})_0 \to \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$ by the formula

$$\mathrm{Res}_b(M) = (\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M_b}).$$

We can also define the parabolic induction functors in the opposite direction. Namely, let $N \in \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$. Then we can define the object $\mathrm{Ind}_b(N) \in \mathcal{O}_c(G, \mathfrak{h})_0$ by the formula

$$\mathrm{Ind}_b(N) = (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0}).$$

Proposition 5.11. (i) *The functors* Ind_b, Res_b *are exact.*

(ii) *One has* $\mathrm{Ind}_b(\mathrm{Res}_b(M)) = E^b(\widehat{M_b})$.

Proof. Part (i) follows from the fact that the functor E^b and the completion functor $\widehat{\ }_b$ are exact (see Proposition 5.10). Part (ii) is straightforward from the definition. □

Theorem 5.12. *The functor* Ind_b *is right adjoint to* Res_b.

Proof. We have

$$\begin{aligned} & \mathrm{Hom}(\mathrm{Res}_b(M), N) = \mathrm{Hom}((\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M_b}), N) = \mathrm{Hom}((E \circ I^{-1} \circ \theta_*)(\widehat{M_b}), \zeta^{-1}(N)) \\ = \ & \mathrm{Hom}((I^{-1} \circ \theta_*)(\widehat{M_b}), \widehat{\zeta^{-1}(N)_0}) = \mathrm{Hom}(\widehat{M_b}, (\theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0})) \\ = \ & \mathrm{Hom}(M, (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0})) = \mathrm{Hom}(M, \mathrm{Ind}_b(N)). \end{aligned}$$

At the end we used Proposition 5.8. □

Then we can obtain the following corollary easily.

Corollary 5.13. *The functor* Res_b *maps projective objects to projective ones, and the functor* Ind_b *maps injective objects to injective ones.*

We can also define functors $\mathrm{res}_\lambda : \mathcal{O}_c(G, \mathfrak{h})_0 \to \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$ and $\mathrm{ind}_\lambda : \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0 \to \mathcal{O}_c(G, \mathfrak{h})_0$, attached to $\lambda \in \mathfrak{h}^{*G'}_{\mathrm{reg}}$, by

$$\mathrm{res}_\lambda := \dagger \circ \mathrm{Res}_\lambda \circ \dagger, \mathrm{ind}_\lambda := \dagger \circ \mathrm{Ind}_\lambda \circ \dagger,$$

where † is as in Subsection 5.3.

Corollary 5.14. *The functors* res_λ, ind_λ *are exact. The functor* ind_λ *is left adjoint to* res_λ. *The functor* ind_λ *maps projective objects to projective ones, and the functor* res_λ *injective objects to injective ones.*

Proof. Easy to see from the definition of the functors and the Theorem 5.12. □

We also have the following proposition, whose proof is straightforward.

Proposition 5.15. *We have*

$$\mathrm{ind}_\lambda(N) = (\mathcal{J} \circ \psi_\lambda^{-1})(N), \quad \text{and} \quad \mathrm{res}_\lambda(M) = (\psi_\lambda \circ E_\lambda)(\widehat{M}),$$

where ψ_λ *is defined in Corollary 5.5.*

5.9. **Some evaluations of the parabolic induction and restriction functors.** For generic c, the category $\mathcal{O}_c(G,\mathfrak{h})$ is semisimple, and naturally equivalent to the category Rep$G$ of finite dimensional representations of G, via the functor $\tau \mapsto M_c(G,\mathfrak{h},\tau)$. (If G is a Coxeter group, the exact set of such c (which are called regular) is known from [GGOR] and [Gy]).

Proposition 5.16. (i) *Suppose that c is generic. Upon the above identification, the functors* Ind_b, ind_λ *and* Res_b, res_λ *go to the usual induction and restriction functors between categories* Rep$G$ *and* RepG'. *In other words, we have*

$$\mathrm{Res}_b(M_c(G,\mathfrak{h},\tau)) = \oplus_{\xi\in\widehat{G'}} n_{\tau\xi} M_c(G',\mathfrak{h}/\mathfrak{h}^{G'},\xi),$$

and

$$\mathrm{Ind}_b(M_c(G',\mathfrak{h}/\mathfrak{h}^{G'},\xi)) = \oplus_{\tau\in\widehat{G}} n_{\tau\xi} M_c(G,\mathfrak{h},\tau),$$

where $n_{\tau\xi}$ is the multiplicity of occurrence of ξ in $\tau|_{G'}$, and similarly for res_λ, ind_λ.

(ii) *The equations of (i) hold at the level of Grothendieck groups for all c.*

Proof. Part (i) is easy for $c = 0$, and is obtained for generic c by a deformation argument. Part (ii) is also obtained by deformation argument, taking into account that the functors Res_b and Ind_b are exact and flat with respect to c. □

Example 5.17. Suppose that $G' = 1$. Then $\mathrm{Res}_b(M)$ is the fiber of M at b, while $\mathrm{Ind}_b(\mathbb{C}) = P_{KZ}$, the object defined in [GGOR], which is projective and injective (see Remark 5.22). This shows that Proposition 5.16 (i) does not hold for special c, as P_{KZ} is not, in general, a direct sum of standard modules.

5.10. **Dependence of the functor Res_b on b.** Let $G' \subset G$ be a parabolic subgroup. In the construction of the functor Res_b, the point b can be made a variable which belongs to the open set $\mathfrak{h}^{G'}_{\mathrm{reg}}$.

Namely, let $\widehat{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ be the formal neighborhood of the locally closed set $\mathfrak{h}^{G'}_{\mathrm{reg}}$ in $\mathfrak{h}$, and let $\pi : \widehat{\mathfrak{h}^{G'}_{\mathrm{reg}}} \to \mathfrak{h}/G$ be the natural map (note that this map is an étale covering of the image with the Galois group $N_G(G')/G'$, where $N_G(G')$ is the normalizer of G' in G). Let $\widehat{H_{1,c}}(G,\mathfrak{h})_{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ be the pullback of the sheaf $H_{1,c,G,\mathfrak{h}}$ under π. We can regard it as a sheaf of algebras over $\mathfrak{h}^{G'}_{\mathrm{reg}}$. Similarly to Theorem 5.4 we have an isomorphism

$$\theta : \widehat{H_{1,c}}(G,\mathfrak{h})_{\mathfrak{h}^{G'}_{\mathrm{reg}}} \to Z(G,G',\widehat{H_{1,c}}(G',\mathfrak{h}/\mathfrak{h}^{G'})_0)\hat{\otimes}\mathcal{D}(\mathfrak{h}^{G'}_{\mathrm{reg}}),$$

where $\mathcal{D}(\mathfrak{h}^{G'}_{\mathrm{reg}})$ is the sheaf of differential operators on $\mathfrak{h}^{G'}_{\mathrm{reg}}$, and $\hat{\otimes}$ is an appropriate completion of the tensor product.

Thus, repeating the construction of Res_b, we can define the functor

$$\mathrm{Res} : \mathcal{O}_c(G,\mathfrak{h})_0 \to \mathcal{O}_c(G',\mathfrak{h}/\mathfrak{h}^{G'})_0 \boxtimes \mathrm{L}oc(\mathfrak{h}^{G'}_{\mathrm{reg}}),$$

where $\mathrm{L}oc(\mathfrak{h}^{G'}_{\mathrm{reg}})$ stands for the category of local systems (i.e. $\mathcal{O}$-coherent $\mathcal{D}$-modules) on $\mathfrak{h}^{G'}_{\mathrm{reg}}$. This functor has the property that Res_b is the fiber of Res at b. Namely, the functor Res is defined by the formula

$$\mathrm{Res}(M) = (E \circ I^{-1} \circ \theta_*)(\widehat{M}_{\mathfrak{h}^{G'}_{\mathrm{reg}}}),$$

where $\widehat{M}_{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ is the restriction of the sheaf M on $\mathfrak{h}$ to the formal neighborhood of $\mathfrak{h}^{G'}_{\mathrm{reg}}$.

Remark 5.18. If G' is the trivial group, the functor Res is just the KZ functor from [GGOR], which we will discuss later. Thus, Res is a relative version of the KZ functor.

Remark 5.19. Note that the object $\mathrm{R}es(M)$ is naturally equivariant under the group $N_G(G')/G'$.

Thus, we see that the functor Res_b does not depend on b, up to an isomorphism. A similar statement is true for the functors Ind_b, res_λ, ind_λ.

Conjecture 5.20. For any $b \in \mathfrak{h}, \lambda \in \mathfrak{h}^*$ such that $G_b = G_\lambda$, we have isomorphisms of functors $\mathrm{Res}_b \cong \mathrm{res}_\lambda$, $\mathrm{Ind}_b \cong \mathrm{ind}_\lambda$.

Remark 5.21. Conjecture 5.20 would imply that Ind_b is left adjoint to Res_b, and that Res_b maps injective objects to injective ones, while Ind_b maps projective objects to projective ones.

Remark 5.22. If b and λ are generic (i.e., $G_b = G_\lambda = 1$) then the conjecture holds. Indeed, in this case the conjecture reduces to showing that we have an isomorphism of functors $\mathrm{F}iber_b(M) \cong \mathrm{F}iber_\lambda(M^\dagger)^*$ $(M \in \mathcal{O}_c(G,\mathfrak{h}))$. Since both functors are exact functors to the category of vector spaces, it suffices to check that $\dim \mathrm{F}iber_b(M) = \dim \mathrm{F}iber_\lambda(M^\dagger)$. But this is true because both dimensions are given by the leading coefficient of the Hilbert polynomial of M (characterizing the growth of M).

It is important to mention, however, that although Res_b is isomorphic to $\mathrm{Res}_{b'}$ if $G_b = G_{b'}$, this isomorphism is not canonical. So let us examine the dependence of Res_b on b a little more carefully.

Theorem 5.16 implies that if c is generic, then

$$\mathrm{Res}(M_c(G,\mathfrak{h},\tau)) = \oplus_\xi M_c(G',\mathfrak{h}/\mathfrak{h}^{G'},\xi) \otimes \mathcal{L}_{\tau\xi},$$

where $\mathcal{L}_{\tau\xi}$ is a local system on $\mathfrak{h}^{G'}_{\mathrm{reg}}$ of rank $n_{\tau\xi}$. Let us characterize the local system $\mathcal{L}_{\tau\xi}$ explicitly.

Proposition 5.23. *The local system $\mathcal{L}_{\tau\xi}$ is given by the connection on the trivial bundle given by the formula*

$$\nabla = \mathrm{d} - \sum_{s\in\mathcal{S}:s\notin G'} \frac{2c_s}{1-\lambda_s}\frac{\mathrm{d}\alpha_s}{\alpha_s}(1-s).$$

with values in $\mathrm{Hom}_{G'}(\xi, \tau|_{G'})$.

Proof. This follows immediately from formula (5.1). □

Definition 5.24. We will call the connection of Proposition 5.23 the parabolic KZ (Knizhnik-Zamolodchikov) connection.

Example 5.25. Let $G = \mathfrak{S}_n$ and $G' = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ with $n_1 + \cdots + n_k = n$. In this case, there is only one parameter c.

Let $\mathfrak{h} = \mathbb{C}^n$ be the permutation representation of G. Then

$$\mathfrak{h}^{G'} = (\mathbb{C}^n)^{G'} = \{v \in \mathfrak{h} | v = (\underbrace{z_1,\ldots,z_1}_{n_1},\underbrace{z_2,\ldots,z_2}_{n_2},\ldots,\underbrace{z_k,\ldots,z_k}_{n_k})\}.$$

Thus, the parabolic KZ connection on the trivial bundle with fiber being a representation τ of $\mathfrak{S}_n$ has the form

$$\mathrm{d} - c \sum_{1\le p<q\le k} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{\mathrm{d}z_p - \mathrm{d}z_q}{z_p - z_q}(1 - s_{ij}).$$

So the differential equations for a flat section $F(z)$ of this bundle have the form

$$\frac{\partial F}{\partial z_p} = c \sum_{q\neq p} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{(1 - s_{ij})F}{z_p - z_q}.$$

So $F(z) = G(z)\prod_{p<q}(z_p - z_q)^{cn_pn_q}$, where the function G satisfies the differential equation

$$\frac{\partial G}{\partial z_p} = -c \sum_{q\neq p} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{s_{ij}G}{z_p - z_q}.$$

Let $\tau = V^{\otimes n}$ where V is a finite dimensional space with $\dim V = N$ (this class of representations contains as summands all irreducible representations of $\mathfrak{S}_n$). Let $V_p = V^{\otimes n_p}$, so that $\tau = V_1 \otimes \cdots \otimes V_k$. Then the equation for G can be written as

$$\frac{\partial G}{\partial z_p} = -c \sum_{q\neq p} \frac{\Omega_{pq}G}{z_p - z_q}, \quad p = 1, \ldots, k,$$

where $\Omega = \sum_{s,t=1}^{N} E_{s,t} \otimes E_{t,s}$ is the Casimir element for $\mathfrak{gl}_N$ ($E_{i,j}$ is the N by N matrix with the only 1 at the (i, j)-th place, and the rest of the entries being 0).

This is nothing but the well known Knizhnik-Zamolodchikov system of equations of the WZW conformal field theory, for the Lie algebra $\mathfrak{gl}_N$, see [EFK]. (Note that the representations V_i are "the most general" in the sense that any irreducible finite dimensional representation of $\mathfrak{gl}_N$ occurs in $V^{\otimes r}$ for some r, up to tensoring with a character.)

This motivates the term "parabolic KZ connection".

5.11. **Supports of modules.** The following two basic propositions are proved in [Gi1], Section 6. We will give different proofs of them, based on the restriction functors.

Proposition 5.26. *Consider the stratification of* $\mathfrak{h}$ *with respect to stabilizers of points in* G*. Then the (set-theoretical) support* Supp$M$ *of any object* M *of* $\mathcal{O}_c(G, \mathfrak{h})$ *in* $\mathfrak{h}$ *is a union of strata of this stratification.*

Proof. This follows immediately from the existence of the flat connection along the set of points b with a fixed stabilizer G' on the bundle $\mathrm{Res}_b(M)$. □

Proposition 5.27. *For any irreducible object* M *in* $\mathcal{O}_c(G, \mathfrak{h})$*,* S*upp*$M/G$ *is an irreducible algebraic variety.*

Proof. Let X be a component of SuppM/G. Let M' be the subspace of elements of M whose restriction to a neighborhood of a generic point of X is zero. It is obvious that M' is an $H_{1,c}(G, \mathfrak{h})$-submodule in M. By definition, it is a proper submodule. Therefore, by the irreducibility of M, we have $M' = 0$. Now let $f \in \mathbb{C}[\mathfrak{h}]^G$ be a function that vanishes on X. Then there exists a positive integer N such that f^N maps M to M', hence acts by zero on M. This implies that Supp$M/G = X$, as desired. □

Propositions 5.26 and 5.27 allow us to attach to every irreducible module $M \in \mathcal{O}_c(G, \mathfrak{h})$, a conjugacy class of parabolic subgroups, $C_M \in \mathrm{Par}(G)$, namely, the conjugacy class of the stabilizer of a generic point of the support of M. Also, for a parabolic subgroup $G' \subset G$, denote by $\mathcal{X}(G')$ the set of points $b \in \mathfrak{h}$ whose stabilizer contains a subgroup conjugate to G'.

The following proposition is immediate.

Proposition 5.28. (i) *Let $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ be irreducible. If b is such that $G_b \in C_M$, then $\mathrm{Res}_b(M)$ is a nonzero finite dimensional module over $H_{1,c}(G_b, \mathfrak{h}/\mathfrak{h}^{G_b})$.*

(ii) *Conversely, let $b \in \mathfrak{h}$, and L be a finite dimensional module $H_{1,c}(G_b, \mathfrak{h}/\mathfrak{h}^{G_b})$. Then the support of $\mathrm{Ind}_b(L)$ in $\mathfrak{h}$ is $\mathcal{X}(G_b)$.*

Let $\mathrm{FD}(G, \mathfrak{h})$ be the set of c for which $H_{1,c}(G, \mathfrak{h})$ admits a finite dimensional representation.

Corollary 5.29. *Let G' be a parabolic subgroup of G. Then $\mathcal{X}(G')$ is the support of some irreducible representation from $\mathcal{O}_c(G, \mathfrak{h})_0$ if and only if $c \in \mathrm{FD}(G', \mathfrak{h}/\mathfrak{h}^{G'})$.*

Proof. Immediate from Proposition 5.28. □

Example 5.30. Let $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$. In this case, the set $\mathrm{Par}(G)$ is the set of partitions of n. Assume that $c = r/m$, $(r, m) = 1$, $2 \le m \le n$. By a result of [BEG], finite dimensional representations of $H_c(G, \mathfrak{h})$ exist if and only if $m = n$. Thus the only possible classes C_M for irreducible modules M have stabilizers $\mathfrak{S}_m \times \cdots \times \mathfrak{S}_m$, i.e., correspond to partitions into parts, where each part is equal to m or 1. So there are $[n/m] + 1$ possible supports for modules, where $[a]$ denotes the integer part of a.

5.12. **Notes.** Our discussion of the geometric approach to rational Cherednik algebras in Section 5.1 follows [E1] and Section 2.2 of [BE]. Our exposition in the other sections follows the corresponding parts of the paper [BE].

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Fall 2009