CHAPTER I

ELEMENTARY DIFFERENTIAL GEOMETRY

§1-§3. When a Euclidean space is stripped of its vector space structure and only its differentiable structure retained, there are many ways of piecing together domains of it in a smooth manner, thereby obtaining a so-called differentiable manifold. Local concepts like a differentiable function and a tangent vector can still be given a meaning whereby the manifold can be viewed "tangentially," that is, through its family of tangent spaces as a curve in the plane is, roughly speaking, determined by its family of tangents. This viewpoint leads to the study of tensor fields, which are important tools in local and global differential geometry. They form an algebra $\mathfrak{D}(M)$, the mixed tensor algebra over the manifold M. The alternate covariant tensor fields (the differential forms) form a submodule $\mathfrak{A}(M)$ of $\mathfrak{D}(M)$ which inherits a multiplication from $\mathfrak{D}(M)$, the exterior multiplication. The resulting algebra is called the Grassmann algebra of M. Through the work of É. Cartan the Grassmann algebra with the exterior differentiation d has become an indispensable tool for dealing with submanifolds, these being analytically described by the zeros of differential forms. Moreover, the pair $(\mathfrak{A}(M), d)$ determines the cohomology of M via de Rham's theorem, which however will not be dealt with here.

§4-§8. The concept of an affine connection was first defined by Levi-Civita for Riemannian manifolds, generalizing significantly the notion of parallelism for Euclidean spaces. On a manifold with a countable basis an affine connection always exists (see the exercises following this chapter). Given an affine connection on a manifold M there is to each curve $\gamma(t)$ in M associated an isomorphism between any two tangent spaces $M_{\gamma(t_1)}$ and $M_{\gamma(t_2)}$. Thus, an affine connection makes it possible to relate tangent spaces at distant points of the manifold. If the tangent vectors of the curve $\gamma(t)$ all correspond under these isomorphisms we have the analog of a straight line, the so-called geodesic. The theory of affine connections mainly amounts to a study of the mappings $\operatorname{Exp}_p : M_p \to M$ under which straight lines (or segments of them) through the origin in the tangent space M_p correspond to geodesics through p in M. Each mapping Exp_p is a diffeomorphism of a neighborhood of 0 in M_p into M, giving the so-called normal coordinates at p.

§1. Manifolds

Let R^m and R^n denote two Euclidean spaces of m and n dimensions, respectively. Let O and O' be open subsets, $O \subset R^m$, $O' \subset R^n$ and suppose φ is a mapping of O into O'. The mapping φ is called *differentiable* if the coordinates $y_j(\varphi(p))$ of $\varphi(p)$ are differentiable (that is, indefinitely differentiable) functions of the coordinates $x_i(p)$, $p \in O$. The mapping φ is called *analytic* if for each point $p \in O$ there exists a neighborhood U of p and n power series P_j $(1 \leqslant j \leqslant n)$ in m variables such that $y_j(\varphi(q)) = P_j(x_1(q) - x_1(p), \ldots, x_m(q) - x_m(p))$ $(1 \leqslant j \leqslant n)$ for $q \in U$. A differentiable mapping $\varphi : O \to O'$ is called *a diffeomorphism* of O onto O' if $\varphi(O) = O'$, φ is one-to-one, and the inverse mapping φ^{-1} is differentiable. In the case when $n = 1$ it is customary to replace the term "mapping" by the term "function."

An analytic function on R^m which vanishes on an open set is identically 0. For differentiable functions the situation is completely different. In fact, if A and B are disjoint subsets of R^m, A compact and B closed, then there exists a differentiable function φ which is identically 1 on A and identically 0 on B. The standard procedure for constructing such a function φ is as follows:

Let $0 < a < b$ and consider the function f on R defined by

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right) & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is differentiable and the same holds for the function

$$F(x) = \int_x^b f(t)\,dt \Big/ \int_a^b f(t)\,dt,$$

which has value 1 for $x \leqslant a$ and 0 for $x \geqslant b$. The function ψ on R^m given by

$$\psi(x_1, \ldots, x_m) = F(x_1^2 + \ldots + x_m^2)$$

is differentiable and has values 1 for $x_1^2 + \ldots + x_m^2 \leqslant a$ and 0 for $x_1^2 + \ldots + x_m^2 \geqslant b$. Let S and S' be two concentric spheres in R^m, S' lying inside S. Starting from ψ we can by means of a linear transformation of R^m construct a differentiable function on R^m with value 1 in the interior of S' and value 0 outside S. Turning now to the sets A and B we can, owing to the compactness of A, find finitely many spheres S_i $(1 \leqslant i \leqslant n)$, such that the corresponding open balls B_i $(1 \leqslant i \leqslant n)$, form a covering of A (that is, $A \subset \bigcup_{i=1}^n B_i$) and such that the closed balls $\bar{B}_i$ $(1 \leqslant i \leqslant n)$ do not intersect B. Each sphere S_i can be shrunk to a concentric sphere S_i' such that the corresponding open balls B_i' still form a covering of A. Let ψ_i be a differentiable function on R^m which is identically 1 on B_i' and identically 0 in the complement of B_i. Then the function

$$\varphi = 1 - (1 - \psi_1)(1 - \psi_2) \ldots (1 - \psi_n)$$

is a differentiable function on R^m which is identically 1 on A and identically 0 on B.

Let M be a topological space. We assume that M satisfies the Hausdorff separation axiom which states that any two different points in M can be separated by disjoint open sets. An *open chart* on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of R^m.

Definition. Let M be a Hausdorff space. A *differentiable structure* on M of dimension m is a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on M where $\varphi_\alpha(U_\alpha)$ is an open subset of $\boldsymbol{R}^m$ such that the following conditions are satisfied:

(M_1) $M = \bigcup_{\alpha \in A} U_\alpha$.

(M_2) For each pair $\alpha, \beta \in A$ the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable mapping of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

(M_3) The collection $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is a maximal family of open charts for which (M_1) and (M_2) hold.

A *differentiable manifold* (or C^∞ *manifold* or simply *manifold*) of dimension m is a Hausdorff space with a differentiable structure of dimension m. If M is a manifold, a *local chart* on M (or a *local coordinate system* on M) is by definition a pair $(U_\alpha, \varphi_\alpha)$ where $\alpha \in A$. If $p \in U_\alpha$ and $\varphi_\alpha(p) = (x_1(p), ..., x_m(p))$, the set U_α is called a *coordinate neighborhood* of p and the numbers $x_i(p)$ are called *local coordinates* of p. The mapping $\varphi_\alpha : q \to (x_1(q), ..., x_m(q))$, $q \in U_\alpha$, is often denoted $\{x_1, ..., x_m\}$.

Remark 1. Condition (M_3) will often be cumbersome to check in specific instances. It is therefore important to note that the condition (M_3) is not essential in the definition of a manifold. In fact, if only (M_1) and (M_2) are satisfied, the family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ can be extended in a unique way to a larger family $\mathfrak{M}$ of open charts such that (M_1), (M_2), and (M_3) are all fulfilled. This is easily seen by defining $\mathfrak{M}$ as the set of all open charts (V, φ) on M satisfying: (1) $\varphi(V)$ is an open set in $\boldsymbol{R}^m$; (2) for each $\alpha \in A$, $\varphi_\alpha \circ \varphi^{-1}$ is a diffeomorphism of $\varphi(V \cap U_\alpha)$ onto $\varphi_\alpha(V \cap U_\alpha)$.

Remark 2. If we let $\boldsymbol{R}^m$ mean a single point for $m = 0$, the preceding definition applies. The manifolds of dimension 0 are then the discrete topological spaces.

Remark 3. A manifold is connected if and only if it is pathwise connected. The proof is left to the reader.

An *analytic structure* of dimension m is defined in a similar fashion. In (M_2) we just replace "differentiable" by "analytic." In this case M is called an *analytic manifold.*

In order to define a *complex manifold* of dimension m we replace $\boldsymbol{R}^m$ in the definition of differentiable manifold by the m-dimensional complex space $\boldsymbol{C}^m$. The condition (M_2) is replaced by the condition that the m coordinates of $\varphi_\beta \circ \varphi_\alpha^{-1}(p)$ should be holomorphic functions of the coordinates of p. Here a function $f(z_1, ..., z_m)$ of m complex variables is called *holomorphic* if at each point $(z_1^0, ..., z_m^0)$ there exists a power series

$$\sum a_{n_1 \dots n_m} (z_1 - z_1^0)^{n_1} \dots (z_m - z_m^0)^{n_m},$$

which converges absolutely to $f(z_1, ..., z_m)$ in a neighborhood of the point.

The manifolds dealt with in the later chapters of this book (mostly

Lie groups and their coset spaces) are analytic manifolds. From Remark 1 it is clear that we can always regard an analytic manifold as a differentiable manifold. It is often convenient to do so because, as pointed out before for $\boldsymbol{R}^m$, the class of differentiable functions is much richer than the class of analytic functions.

Let f be a real-valued function on a C^∞ manifold M. The function f is called *differentiable* at a point $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that the composite function $f \circ \varphi_\alpha^{-1}$ is a differentiable function on $\varphi_\alpha(U_\alpha)$. The function f is called *differentiable* if it is differentiable at each point $p \in M$. If M is analytic, the function f is said to be *analytic* at $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that $f \circ \varphi_\alpha^{-1}$ is an analytic function on the set $\varphi_\alpha(U_\alpha)$.

Let M be a differentiable manifold and let $\mathcal{F}$ denote the set of differential functions on M.

We shall often write $C^\infty(M)$ instead of $\mathfrak{F}$ and will sometimes denote by $C^\infty(p)$ the set of functions on M which are differentiable at p. The set $C^\infty(M)$ is an algebra over $\boldsymbol{R}$, the operations being

$$(\lambda f)(p) = \lambda f(p),$$

$$(f+g)(p) = f(p) + g(p),$$

$$(fg)(p) = f(p)g(p)$$

for $\lambda \in \boldsymbol{R}$, $p \in M$, $f, g \in C^\infty(M)$.

Lemma 1.2. *Let C be a compact subset of a manifold M and let V be an open subset of M containing C. Then there exists a function $\psi \in C^\infty(M)$ which is identically 1 on C, identically 0 outside V.*

This lemma has already been established in the case $M = \boldsymbol{R}^m$. We shall now show that the general case presents no additional difficulties.

Let $(U_\alpha, \varphi_\alpha)$ be a local chart on M and S a compact subset of U_α. There exists a differentiable function f on $\varphi_\alpha(U_\alpha)$ such that f is identically 1 on $\varphi_\alpha(S)$ and has compact support contained in $\varphi_\alpha(U_\alpha)$. The function F on M given by

$$F(q) = \begin{cases} f(\varphi_\alpha(q)) & \text{if } q \in U_\alpha, \\ 0 & \text{otherwise} \end{cases}$$

is a differentiable function on M which is identically 1 on S and identically 0 outside U_α. Since C is compact and V open, there exist finitely many coordinate neighborhoods $U_1, \ldots, U_n$ and compact sets $S_1, \ldots, S_n$ such that

$$C \subset \bigcup_1^n S_i, \qquad S_i \subset U_i$$

$$\left(\bigcup_1^n U_i\right) \subset V.$$

As shown previously, there exists a function $F_i \in C^\infty(M)$ which is identically 1 on S_i and identically 0 outside U_i. The function

$$\psi = 1 - (1 - F_1)(1 - F_2) \ldots (1 - F_n)$$

belongs to $C^\infty(M)$, is identically 1 on C and identically 0 outside V.

Let M be a C^∞ manifold and $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ a collection satisfying (M_1), (M_2), and (M_3). If U is an open subset of M, U can be given a differentiable structure by means of the open charts $(V_\alpha, \psi_\alpha)_{\alpha \in A}$ where $V_\alpha = U \cap U_\alpha$ and ψ_α is the restriction of φ_α to V_α. With this structure, U is called an *open submanifold* of M. In particular, since M is locally connected, each connected component of M is an open submanifold of M.

Let M and N be two manifolds of dimension m and n, respectively. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ be collections of open charts on M and N, respectively, such that the conditions (M_1), (M_2), and (M_3) are satisfied. For $\alpha \in A$, $\beta \in B$, let $\varphi_\alpha \times \psi_\beta$ denote the mapping $(p, q) \to (\varphi_\alpha(p), \psi_\beta(q))$ of the product set $U_\alpha \times V_\beta$ into R^{m+n}. Then the collection $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)_{\alpha \in A, \beta \in B}$ of open charts on the product space $M \times N$ satisfies (M_1) and (M_2) so by Remark 1, $M \times N$ can be turned into a manifold the *product* of M and N.

An immediate consequence of Lemma 1.2 is the following fact which will often be used: Let V be an open submanifold of M, f a function in $C^\infty(V)$, and p a point in V. Then there exists a function $\tilde{f} \in C^\infty(M)$ and an open neighborhood N, $p \in N \subset V$ such that f and $\tilde{f}$ agree on N.

1. Vector Fields and 1-Forms

Let A be an algebra over a field K. A *derivation* of A is a mapping $D: A \to A$ such that

(i) $D(\alpha f + \beta g) = \alpha Df + \beta Dg$ for $\alpha, \beta \in K$, $f, g \in A$;
(ii) $D(fg) = f(Dg) + (Df)g$ for $f, g \in A$.

Definition. A *vector field* X on a C^∞ manifold is a derivation of the algebra $C^\infty(M)$.

Let $\mathfrak{D}^1$ (or $\mathfrak{D}^1(M)$) denote the set of all vector fields on M. If $f \in C^\infty(M)$ and $X, Y \in \mathfrak{D}^1(M)$, then fX and $X + Y$ denote the vector fields

$$fX : g \to f(Xg), \qquad g \in C^\infty(M),$$
$$X + Y : g \to Xg + Yg, \qquad g \in C^\infty(M).$$

This turns $\mathfrak{D}^1(M)$ into a module over the ring $\mathfrak{F} = C^\infty(M)$. If X, $Y \in \mathfrak{D}^1(M)$, then $XY - YX$ is also a derivation of $C^\infty(M)$ and is denoted by the bracket $[X, Y]$. As is customary we shall often write $\theta(X)\, Y = [X, Y]$. The operator $\theta(X)$ is called the *Lie derivative* with respect to X. The bracket satisfies the *Jacobi identity* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ or, otherwise written $\theta(X)([Y, Z]) = [\theta(X)\, Y, Z] + [Y, \theta(X)\, Z]$.

It is immediate from (ii) that if f is constant and $X \in \mathfrak{D}^1$, then $Xf = 0$. Suppose now that a function $g \in C^\infty(M)$ vanishes on an open subset $V \subset M$. Let p be an arbitrary point in V. According to Lemma 1.2 there exists a function $f \in C^\infty(M)$ such that $f(p) = 0$, and $f = 1$ outside V. Then $g = fg$ so

$$Xg = f(Xg) + g(Xf),$$

which shows that Xg vanishes at p. Since p was arbitrary, $Xg = 0$ on V. We can now define Xf on V for every function $f \in C^\infty(V)$. If $p \in V$, select $\tilde{f} \in C^\infty(M)$ such that f and $\tilde{f}$ coincide in a neighborhood of p and put $(Xf)(p) = (X\tilde{f})(p)$. The consideration above shows that this is a valid definition, that is, independent of the choice of $\tilde{f}$. This shows that a vector field on a manifold induces a vector field on any open submanifold.

On the other hand, let Z be a vector field on an open submanifold $V \subset M$ and p a point in V. Then there exists a vector field $\tilde{Z}$ on M and an open neighborhood N, $p \in N \subset V$ such that $\tilde{Z}$ and Z induce the same vector field on N. In fact, let C be any compact neighborhood of p contained in V and let N be the interior of C. Choose $\psi \in C^\infty(M)$ of compact support contained in V such that $\psi = 1$ on C. For any $g \in C^\infty(M)$, let g_V denote its restriction to V and define $\tilde{Z}g$ by

$$\tilde{Z}(g)\,(q) = \begin{cases} \psi(q)\,(Zg_V)\,(q) & \text{for } q \in V, \\ 0 & \text{if } q \notin V. \end{cases}$$

Then $g \to \tilde{Z}g$ is the desired vector field on M.

Now, let (U, φ) be a local chart on M, X a vector field on U, and let p be an arbitrary point in U. We put $\varphi(q) = (x_1(q), \ldots, x_m(q))$ $(q \in U)$, and $f^* = f \circ \varphi^{-1}$ for $f \in C^\infty(M)$. Let V be an open subset of U such that $\varphi(V)$ is an open ball in R^m with center $\varphi(p) = (a_1, \ldots, a_m)$. If $(x_1, \ldots, x_m) \in \varphi(V)$, we have

$$\begin{aligned} f^*(x_1, \ldots, x_m) & \\ &= f^*(a_1, \ldots, a_m) + \int_0^1 \frac{\partial}{\partial t} f^*(a_1 + t(x_1 - a_1), \ldots, a_m + t(x_m - a_m))\, dt \\ &= f^*(a_1, \ldots, a_m) + \sum_{j=1}^{m} (x_j - a_j) \int_0^1 f_j^*(a_1 + t(x_1 - a_1), \ldots, a_m + t(x_m - a_m))dt. \end{aligned}$$

(Here f_j^* denotes the partial derivative of f^* with respect to the jth argument.) Transferring this relation back to M we obtain

$$f(q) = f(p) + \sum_{i=1}^{m} (x_i(q) - x_i(p))\, g_i(q) \qquad (q \in V), \tag{1}$$

where $g_i \in C^\infty(V)$ $(1 \leqslant i \leqslant m)$, and

$$g_i(p) = \left(\frac{\partial f^*}{\partial x_i}\right)_{\varphi(p)}.$$

It follows that

$$(Xf)(p) = \sum_{i=1}^{m} \left(\frac{\partial f^*}{\partial x_i}\right)_{\varphi(p)} (Xx_i)(p) \qquad \text{for } p \in U. \tag{2}$$

The mapping $f \to (\partial f^*/\partial x_i) \circ \varphi$ $(f \in C^\infty(U))$ is a vector field on U and is denoted $\partial/\partial x_i$. We write $\partial f/\partial x_i$ instead of $\partial/\partial x_i(f)$. Now, by (2)

$$X = \sum_{i=1}^{m} (Xx_i) \frac{\partial}{\partial x_i} \qquad \text{on } U. \tag{3}$$

Thus, $\partial/\partial x_i$ $(1 \leqslant i \leqslant m)$ is a basis of the module $\mathfrak{D}^1(U)$.

For $p \in M$ and $X \in \mathfrak{D}^1$, let X_p denote the linear mapping X_p: $f \to (Xf)(p)$ of $C^\infty(p)$ into R. The set $\{X_p : X \in \mathfrak{D}^1(M)\}$ is called the *tangent space* to M at p; it will be denoted by $\mathfrak{D}^1(p)$ or M_p and its elements are called the *tangent vectors* to M at p. Relation (2) shows that M_p is a vector space over R spanned by the m linearly independent vectors

$$e_i : f \to \left(\frac{\partial f^*}{\partial x_i}\right)_{\varphi(p)}, \qquad f \in C^\infty(M).$$

This tangent vector e_i will often be denoted by $(\partial/\partial x_i)_p$. A linear mapping $L : C^\infty(p) \to R$ is a tangent vector to M at p if and only if the condition

$L(fg) = f(p)\, L(g) + g(p)\, L(f)$ is satisfied for all $f, g \in C^\infty(p)$. In fact, the necessity of the condition is obvious and the sufficiency is a simple consequence of (1). Thus, a vector field X on M can be identified with a collection $X_p (p \in M)$ of tangent vectors to M with the property that for each $f \in C^\infty(M)$ the function $p \to X_p f$ is differentiable.

Suppose the manifold M is analytic. The vector field X on M is then called *analytic* at p if Xf is analytic at p whenever f is analytic at p.

Remark. Let V be a finite-dimensional vector space over $\boldsymbol{R}$. If $X_1, ..., X_n$ is any basis of V, the mapping $\sum_{i=1}^n x_i X_i \to (x_1, ..., x_n)$ is an open chart valid on the entire V. The resulting differentiable structure is independent of the choice of basis. If $X \in V$, the tangent space V_X is identified with V itself by the formula

$$(Yf)(X) = \left\{\frac{d}{dt} f(X + tY)\right\}_{t=0}, \qquad f \in C^\infty(V),$$

which to each $Y \in V$ assigns a tangent vector to V at X.

§ 3. Mappings

1. The Interpretation of the Jacobian

Let M and N be C^∞ manifolds and Φ a mapping of M into N. Let $p \in M$. The mapping Φ is called *differentiable at p* if $g \circ \Phi \in C^\infty(p)$ for each $g \in C^\infty(\Phi(p))$. The mapping Φ is called *differentiable* if it is differentiable at each $p \in M$. Similarly *analytic* mappings are defined. Let $\psi : q \to (x_1(q), ..., x_m(q))$ be a system of coordinates on a neighborhood U of $p \in M$ and $\psi' : r \to (y_1(r), ..., y_n(r))$ a system of coordinates on a neighborhood U' of $\Phi(p)$ in N. Assume $\Phi(U) \subset U'$. The mapping $\psi' \circ \Phi \circ \psi^{-1}$ of $\psi(U)$ into $\psi'(U')$ is given by a system of n functions

$$y_j = \varphi_j(x_1, ..., x_m) \qquad (1 \leqslant j \leqslant n), \tag{1}$$

which we call the *expression* of Φ in coordinates. The mapping Φ is differentiable at p if and only if the functions φ_i have partial derivatives of all orders in some fixed neighborhood of $(x_1(p), ..., x_m(p))$.

The mapping Φ is called a *diffeomorphism* of M onto N if Φ is a one-to-one differentiable mapping of M onto N and Φ^{-1} is differentiable. If in addition M, N, Φ, and Φ^{-1} are analytic, Φ is called an *analytic diffeomorphism.*

If Φ is differentiable at $p \in M$ and $A \in M_p$, then the linear mapping $B : C^\infty(\Phi(p)) \to \boldsymbol{R}$ given by $B(g) = A(g \circ \Phi)$ for $g \in C^\infty(\Phi(p))$ is a tangent vector to N at $\Phi(p)$. The mapping $A \to B$ of M_p into $N_{\Phi(p)}$ is denoted $d\Phi_p$ (or just Φ_p) and is called the *differential* of Φ at p. We have seen that the vectors

$$e_i : f \to \left(\frac{\partial f^*}{\partial x_i}\right)_{\psi(p)} \qquad (1 \leqslant i \leqslant m), \qquad f^* = f \circ \psi^{-1},$$

$$\bar{e}_j : g \to \left(\frac{\partial g^*}{\partial y_j}\right)_{\psi'(\Phi(p))} \qquad (1 \leqslant j \leqslant n), \qquad g^* = g \circ (\psi')^{-1}$$

form a basis of M_p and $N_{\Phi(p)}$, respectively. Then

$$d\Phi_p(e_i)\, g = e_i(g \circ \Phi) = \left(\frac{\partial(g \circ \Phi)^*}{\partial x_i}\right)_{\varphi(p)}.$$

But

$$(g \circ \Phi)^*(x_1, \ldots, x_m) = g^*(y_1, \ldots, y_n),$$

where $y_j = \varphi_j(x_1, \ldots, x_m)$ $(1 \leqslant j \leqslant n)$. Hence

$$d\Phi_p(e_i) = \sum_{j=1}^{n} \left(\frac{\partial \varphi_j}{\partial x_i}\right)_{\varphi(p)} \bar{e}_j. \tag{2}$$

This shows that if we use the bases e_i $(1 \leqslant i \leqslant m)$, $\bar{e}_j$ $(1 \leqslant j \leqslant n)$ to express the linear transformation $d\Phi_p$ in matrix form, then the matrix we obtain is just the Jacobian of the system (1). From a standard theorem on the Jacobian (the inverse function theorem), we can conclude:

Proposition 3.1. *If $d\Phi_p$ is an isomorphism of M_p onto $N_{\Phi(p)}$, then there exist open submanifolds $U \subset M$ and $V \subset N$ such that $p \in U$ and Φ is a diffeomorphism of U onto V.*

Definition.

Let M and N be differentiable (or analytic) manifolds.

(a) A mapping $\Phi : M \to N$ is called *regular* at $p \in M$ if Φ is differentiable (analytic) at $p \in M$ and $d\Phi_p$ is a one-to-one mapping of M_p into $N_{\Phi(p)}$.

(b) M is called a *submanifold* of N if (1) $M \subset N$ (set theoretically); (2) the identity mapping I of M into N is regular at each point of M.

For example, the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ is a submanifold of R^3 and a topological subspace as well. However, a submanifold M of a manifold N is not necessarily a topological subspace of N. For example, let N be a torus and let M be a curve on N without double points, dense in N (Chapter II, §2). Proposition 3.1 shows that a submanifold M of a manifold N is an open submanifold of N if and only if $\dim M = \dim N$.

Proposition 3.2. *Let M be a submanifold of a manifold N and let $p \in M$. Then there exists a coordinate system $\{x_1, \ldots, x_n\}$ valid on an open neighborhood V of p in N such that $x_1(p) = \ldots = x_n(p) = 0$ and such that the set*

$$U = \{q \in V : x_j(q) = 0 \text{ for } m + 1 \leqslant j \leqslant n\}$$

together with the restrictions of $(x_1, \ldots, x_m)$ to U form a local chart on M containing p.

Proof. Let $\{y_1, \ldots, y_m\}$ and $\{z_1, \ldots, z_n\}$ be coordinate systems valid on open neighborhoods of p in M and N, respectively, such that $y_i(p) = z_j(p) = 0$, $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$. The expression of the identity mapping $I: M \to N$ is (near p) given by a system of functions $z_j = \varphi_j(y_1, \ldots, y_m)$, $1 \leqslant j \leqslant n$. The Jacobian matrix $(\partial\varphi_j/\partial y_i)$ of this system has rank m at p since I is regular at p. Without loss of generality we may assume that the square matrix $(\partial\varphi_j/\partial y_i)_{1 \leqslant i, j \leqslant m}$ has determinant $\neq 0$ at p. In a neighborhood of $(0, \ldots, 0)$ we have therefore $y_i = \psi_i(z_1, \ldots, z_m)$, $1 \leqslant i \leqslant m$, where each ψ_i is a differentiable function. If we now put

$$x_i = z_i, \qquad 1 \leqslant i \leqslant m,$$

$$x_j = z_j - \varphi_j(\psi_1(z_1, \ldots, z_m), \ldots, \psi_m(z_1, \ldots, z_m)), \qquad m+1 \leqslant j \leqslant n,$$

it is clear that

$$\det\left(\frac{\partial x_i}{\partial y_l}\right)_{1 \leqslant i,\, l \leqslant m} \neq 0, \qquad \det\left(\frac{\partial x_j}{\partial z_k}\right)_{1 \leqslant j,\, k \leqslant n} \neq 0.$$

Therefore $\{x_1, \ldots, x_n\}$ gives the desired coordinate system.

2. Transformation of Vector Fields

Let M and N be C^∞ manifolds and Φ a differentiable mapping of M into N. Let X and Y be vector fields on M and N, respectively; X and Y are called *Φ-related* if

$$d\Phi_p(X_p) = Y_{\Phi(p)} \qquad \text{for all } p \in M. \tag{3}$$

It is easy to see that (3) is equivalent to

$$(Yf) \circ \Phi = X(f \circ \Phi) \qquad \text{for all } f \in C^\infty(N). \tag{4}$$

It is convenient to write $d\Phi \cdot X = Y$ or $X^\Phi = Y$ instead of (3).

Proposition 3.3.

(i) *Suppose X_i and Y_i are Φ-related, $(i = 1, 2)$. Then*

$$[X_1, X_2] \textit{ and } [Y_1, Y_2] \textit{ are } \Phi\textit{-related.}$$

(ii) *Suppose Φ is a diffeomorphism of M onto itself and put $f^\Phi = f \circ \Phi^{-1}$ for $f \in C^\infty(M)$. Then if $X \in \mathfrak{D}^1(M)$,*

$$(fX)^\Phi = f^\Phi X^\Phi, \qquad (Xf)^\Phi = X^\Phi f^\Phi.$$

Proof. From (4) we have $(Y_1(Y_2 f)) \circ \Phi = X_1(Y_2 f \circ \Phi) = X_1(X_2(f \circ \Phi))$, so (i) follows. The last relation in (ii) is also an immediate consequence of (4). As to the first one, we have for $g \in C^\infty(M)$

$$((fX)^\Phi g) \circ \Phi = (fX)(g \circ \Phi) = f((X^\Phi g) \circ \Phi),$$

so

$$(fX)^\Phi g = f^\Phi (X^\Phi g).$$

Remark. Since $X^{\Phi}f = (Xf^{\Phi^{-1}})^{\Phi}$ it is natural to make the following definition. Let Φ be a diffeomorphism of M onto M and A a mapping of $C^{\infty}(M)$ into itself. The mapping A^{Φ} is defined by $A^{\Phi}f = (Af^{\Phi^{-1}})^{\Phi}$ for $f \in C^{\infty}(M)$. We also write $[Af](p)$ for the value of the function Af at $p \in M$. If Φ and Ψ are two diffeomorphisms of M, then $f^{\Phi\Psi} = (f^{\Psi})^{\Phi}$ and $A^{\Phi\Psi} = (A^{\Psi})^{\Phi}$.

Let M be a differentiable manifold, S a submanifold. Let $m = \dim M$, $s = \dim S$. A curve in S is of course a curve in M, but a curve in M contained in S is not necessarily a curve in S, because it may not even be continuous. However, we have:

Lemma 3.4 *Let φ be a differentiable mapping of a manifold V into the manifold M such that $\varphi(V)$ is contained in the submanifold S. If the mapping $\varphi : V \to S$ is continuous it is also differentiable.*

Let $p \in V$. In view of Prop. 3.2, there exists a coordinate system $\{x_1, \ldots, x_m\}$ valid on an open neighborhood N of $\varphi(p)$ in M such that the set

$$N_S = \{r \in N : x_j(r) = 0 \text{ for } s < j \leqslant m\}$$

together with the restrictions of $(x_1, \ldots, x_s)$ to N_S form a local chart on S containing $\varphi(p)$. By the continuity of φ there exists a local chart (W, ψ) around p such that $\varphi(W) \subset N_S$. The coordinates $x_j(\varphi(q))$ $(1 \leqslant j \leqslant m)$ depend differentiably on the coordinates of $q \in W$. In particular, this holds for the coordinates $x_j(\varphi(q))$ $(1 \leqslant j \leqslant s)$ so the mapping $\varphi : V \to S$ is differentiable.

As an immediate consequence of this lemma we have the following statement: Suppose that V and S are submanifolds of M and $V \subset S$. If S has the relative topology of M, then V is a submanifold of S.

§ 4. Affine Connections

Definition. An *affine connection* on a manifold M is a rule ∇ which assigns to each $X \in \mathfrak{D}^1(M)$ a linear mapping ∇_X of the vector space $\mathfrak{D}^1(M)$ into itself satisfying the following two conditions:

$$(\nabla_1) \qquad \nabla_{fX+gY} = f\nabla_X + g\nabla_Y;$$

$$(\nabla_2) \qquad \nabla_X(fY) = f\nabla_X(Y) + (Xf)\,Y$$

for $f, g \in C^{\infty}(M)$, $X, Y \in \mathfrak{D}^1(M)$. The operator ∇_X is called *covariant differentiation* with respect to X. For a motivation see Exercises.

Lemma 4.1. *Suppose M has the affine connection $X \to \nabla_X$ and let U be an open submanifold of M. Let $X, Y \in \mathfrak{D}^1(M)$. If X or Y vanishes identically on U, then so does $\nabla_X(Y)$.*

Proof. Suppose Y vanishes on U. Let $p \in U$ and $g \in C^{\infty}(M)$. To prove that $(\nabla_X(Y)g)(p) = 0$, we select $f \in C^{\infty}(M)$ such that $f(p) = 0$ and $f = 1$ outside U (Lemma 1.2). Then $fY = Y$ and

$$\nabla_X(Y)g = \nabla_X(fY)g = (Xf)(Yg) + f(\nabla_X(Y)g)$$

which vanishes at p. The statement about X follows similarly.

An affine connection ∇ on M induces an affine connection ∇_U on an arbitrary open submanifold U of M. In fact, let X, Y be two vector fields on U. For each $p \in U$ there exist vector fields X', Y' on M which agree with X and Y in an open neighborhood V of p. We then put $((\nabla_U)_X(Y))_q = (\nabla_{X'}(Y'))_q$ for $q \in V$. By Lemma 4.1, the right-hand side of this equation is independent of the choice of X', Y'. It follows immediately that the rule ∇_U: $X \to (\nabla_U)_X$ $(X \in \mathfrak{D}^1(U))$ is an affine connection on U.

In particular, suppose U is a coordinate neighborhood where a coordinate system $\varphi : q \to (x_1(q), \ldots, x_m(q))$ is valid. For simplicity, we write ∇_i instead of $(\nabla_U)_{\partial/\partial x_i}$. We define the functions $\Gamma_{ij}{}^k$ on U by

$$\nabla_i \left(\frac{\partial}{\partial x_j}\right) = \sum_k \Gamma_{ij}{}^k \frac{\partial}{\partial x_k}. \tag{1}$$

For simplicity of notation we write also $\Gamma_{ij}{}^k$ for the function $\Gamma_{ij}{}^k \circ \varphi^{-1}$. If $\{y_1, \ldots, y_m\}$ is another coordinate system valid on U, we get another set of functions $\Gamma'_{\alpha\beta}{}^\gamma$ by

$$\nabla_\alpha \left(\frac{\partial}{\partial y_\beta}\right) = \sum_\gamma \Gamma'_{\alpha\beta}{}^\gamma \frac{\partial}{\partial y_\gamma}.$$

Using the axioms ∇_1 and ∇_2 we find easily

$$\Gamma'_{\alpha\beta}{}^\gamma = \sum_{i,j,k} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial x_j}{\partial y_\beta} \frac{\partial y_\gamma}{\partial x_k} \Gamma_{ij}{}^k + \sum_j \frac{\partial^2 x_j}{\partial y_\alpha \partial y_\beta} \frac{\partial y_\gamma}{\partial x_j}. \tag{2}$$

On the other hand, suppose there is given a covering of a manifold M by open coordinate neighborhoods U and in each neighborhood a system of functions $\Gamma_{ij}{}^k$ such that (2) holds whenever two of these neighborhoods overlap. Then we can define ∇_i by (1) and thus we get an affine connection ∇_U in each coordinate neighborhood U. We finally define an affine connection $\tilde{\nabla}$ on M as follows: Let X, $Y \in \mathfrak{D}^1(M)$ and $p \in M$. If U is a coordinate neighborhood containing p, let

$$(\tilde{\nabla}_X(Y))_p = ((\nabla_U)_{X_1}(Y_1))_p$$

if X_1 and Y_1 are the vector fields on U induced by X and Y, respectively. Then $\tilde{\nabla}$ is an affine connection on M which on each coordinate neighborhood U induces the connection ∇_U.

Lemma 4.2. *Let X, $Y \in \mathfrak{D}^1(M)$. If X vanishes at a point p in M, then so does $\nabla_X(Y)$.*

Let $\{x_1, \ldots, x_m\}$ be a coordinate system valid on an open neighborhood U of p. On the set U we have $X = \sum_i f_i(\partial/\partial x_i)$ where $f_i \in C^\infty(U)$ and $f_i(p) = 0$, $(1 \leqslant i \leqslant m)$. Using Lemma 4.1 we find $(\nabla_X(Y))_p = \sum_i f_i(p)\,(\nabla_i(Y))_p = 0$.

Remark. Thus if $v \in M_p$, $\nabla_v(Y)$ is a well-defined vector in M_p.

Definition. Suppose ∇ is an affine connection on M and that Φ is a diffeomorphism of M. A new affine connection ∇' can be defined on M by

$$\nabla'_X(Y) = (\nabla_{X^\Phi}(Y^\Phi))^{\Phi^{-1}}, \qquad X, Y \in \mathfrak{D}^1(M).$$

That ∇' is indeed an affine connection on M is best seen from Prop. 3.3.

The affine connection ∇ is called *invariant* under Φ if $\nabla' = \nabla$. In this case Φ is called an *affine transformation* of M. Similarly one can define an affine transformation of one manifold onto another.

§ 5. Parallelism

Let M be a C^∞ manifold. A *curve* in M is a regular mapping of an open interval $I \subset R$ into M. The restriction of a curve to a closed subinterval is called a *curve segment*. The curve segment is called finite if the interval is finite.

Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in M. Differentiation with respect to the parameter will often be denoted by a dot $(\cdot)$. In particular, $\dot\gamma(t)$ stands for the tangent vector $d\gamma(d/dt)_t$. Suppose now that to each $t \in I$ is associated a vector $Y(t) \in M_{\gamma(t)}$. Assuming $Y(t)$ to vary differentiably with t, we shall now define what it means for the family $Y(t)$ to be parallel with respect to γ. Let J be a compact subinterval of I such that the finite curve segment $\gamma_J : t \to \gamma(t)$ $(t \in J)$ has no double points and such that $\gamma(J)$ is contained in a coordinate neighborhood U. Owing to the regularity of γ each point of I is contained in such an interval J with nonempty interior. Let $\{x_1, \ldots, x_m\}$ be a coordinate system on U.

Lemma 5.1. *Let $t \to \gamma(t)$ $(t \in I)$ be a curve in a manifold M. Let $t_0 \in I$ and g a smooth function on a neighborhood of t_0 in I. Then $\exists$ open interval I_{t_0} around t_0 in I and a function $G \in \mathcal{C}^\infty(M)$ such that*

$$G(\gamma(t)) = g(t) \quad t \in I_{t_0}.$$

Proof:

Let $\{x_1, \ldots, x_m\}$ be a coordinate system on a neighborhood of $\gamma(t_0)$ in M. There exists an index i such that $t \to x_i(\gamma(t))$ has a nonzero derivative at $t = t_0$. Thus $\exists$ smooth function η_i of 1-variable smooth in a neighborhood of $x_i(\gamma(t_0))$ in $\mathbf{R}$ such that $t = \eta_i(x_i(\gamma(t)))$ for all t in a neighborhood of t_0. The function

$$q \to g(\eta_i(x_i(q)))$$

is defined and smooth in a neighborhood of $\gamma(t_0)$ in M. In a smaller neighborhood it coincides with a function $G \in \mathcal{C}^\infty(M)$. But then

$$G(\gamma(t)) = g(\eta_i(x_i(\gamma(t)))) = g(t)$$

for t in an interval around t_0.

We put $X(t) = \dot{\gamma}(t)$ $(t \in I)$. Using Lemma 5.1 it is easy to see that there exist vector fields $X, Y \in \mathfrak{D}^1(M)$ such that ($Y(t)$ being as before)

$$X_{\gamma(t)} = X(t), \qquad Y_{\gamma(t)} = Y(t) \qquad (t \in I_{t_0})$$

Given an affine connection ∇ on M, the family $Y(t)$ $(t \in J)$ is said to be *parallel* with respect to γ_J (or parallel along γ_J) if

$$\nabla_X(Y)_{\gamma(t)} = 0 \qquad \text{for all } t \in I_{t_0} \tag{1}$$

To show that this definition is independent of the choice of X and Y, we express (1) in the coordinates $\{x_1, \ldots, x_m\}$. There exist functions X^i, Y^j $(1 \leqslant i, j \leqslant m)$ on U such that

$$X = \sum_i X^i \frac{\partial}{\partial x_i}, \qquad Y = \sum_j Y^j \frac{\partial}{\partial x_j} \qquad \text{on } U.$$

For simplicity we put $x_i(t) = x_i(\gamma(t))$, $X^i(t) = X^i(\gamma(t))$, and $Y^i(t) = Y^i(\gamma(t))$ $(t \in I_{t_0})(1 \leqslant i \leqslant m)$. Then $X^i(t) = \dot{x}_i(t)$ and since

$$\nabla_X(Y) = \sum_k \Big(\sum_i X^i \frac{\partial Y^k}{\partial x_i} + \sum_{i,j} X^i Y^j \Gamma_{ij}{}^k\Big) \frac{\partial}{\partial x_k} \qquad \text{on } U$$

we obtain

$$\frac{dY^k}{dt} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt} Y^j = 0 \qquad (t \in J). \tag{2}$$

This equation involves X and Y only through their values *on* the curve. Consequently, condition (1) for parallelism is independent of the choice of X and Y. It is now obvious how to define parallelism with respect to any finite curve segment γ_J and finally with respect to the entire curve γ.

Definition. Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in M. The curve γ is called a *geodesic* if the family of tangent vectors $\dot{\gamma}(t)$ is parallel with respect to γ. A geodesic γ is called maximal if it is not a proper restriction of any geodesic.

Suppose γ_J is a finite geodesic segment without double points contained in a coordinate neighborhood U where the coordinates $\{x_1, ..., x_m\}$ are valid. Then (2) implies

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0 \qquad (t \in J). \tag{3}$$

If we change the parameter on the geodesic and put $t = f(s)$, $(f'(s) \neq 0)$, then we get a new curve $s \to \gamma_J(f(s))$. This curve is a geodesic if and only if f is a linear function, as (3) shows.

Proposition 5.2. *Let p, q be two points in M and γ a curve segment from p to q. The parallelism τ with respect to γ induces an isomorphism of M_p onto M_q.*

Proof. Without loss of generality we may assume that γ has no double points and lies in a coordinate neighborhood U. Let $\{x_1, ..., x_m\}$ be a system of coordinates on U. Suppose the curve segment γ is given by the mapping $t \to \gamma(t)$ $(a \leqslant t \leqslant b)$ such that $\gamma(a) = p$, $\gamma(b) = q$. As before we put $x_i(t) = x_i(\gamma(t))$ $(a \leqslant t \leqslant b)$ $(1 \leqslant i \leqslant m)$.

Consider the system (2). From the theory of systems of ordinary, linear differential equations of first order we can conclude:

There exist m functions $\varphi_i(t, y_1, ..., y_m)$ $(1 \leqslant i \leqslant m)$ defined and differentiable† for $a \leqslant t \leqslant b$, $-\infty < y_i < \infty$ such that

(i) For each m-tuple $(y_1, ..., y_m)$, the functions $Y^i(t) = \varphi_i(t, y_1, ..., y_m)$ satisfy the system (2).

(ii) $\varphi_i(a, y_1, ..., y_m) = y_i \qquad (1 \leqslant i \leqslant m)$.

The functions φ_i are uniquely determined by these properties.

The properties (i) and (ii) show that the family of vectors $Y(t) = \Sigma_i Y^i(t)\,(\partial/\partial x_i)$ $(a \leqslant t \leqslant b)$ is parallel with respect to γ and that $Y(a) = \Sigma_i y_i(\partial/\partial x_i)_p$. The mapping $Y(a) \to Y(b)$ is a linear mapping of M_p into M_q since the functions φ_i are linear in the variables $y_1, ..., y_m$. This mapping is one-to-one owing to the uniqueness of the functions φ_i. Consequently, it is an isomorphism.

Proposition 5.3. *Let M be a differentiable manifold with an affine connection. Let p be any point in M and let $X \neq 0$ in M_p. Then there exists a unique maximal geodesic $t \to \gamma(t)$ in M such that*

$$\gamma(0) = p, \qquad \dot\gamma(0) = X. \tag{4}$$

† A function on a closed interval I is called differentiable on I if it is extendable to a differentiable function on some open interval containing I.

Proof. Let $\varphi : q \to (x_1(q), ..., x_m(q))$ be a system of coördinates on a neighborhood U of p such that $\varphi(U)$ is a cube $\{(x_1, ..., x_m) : | x_i | < c\}$ and $\varphi(p) = 0$. Then X can be written $X = \sum_i \alpha_i(\partial/\partial x_i)_p$ where $\alpha_i \in \boldsymbol{R}$. We consider the system of differential equations

$$\frac{dx_i}{dt} = z_i \qquad (1 \leqslant i \leqslant m), \tag{5}$$

$$\frac{dz_k}{dt} = -\sum_{i,j=1}^{m} \Gamma_{ij}{}^k (x_1, ..., x_m)\, z_i z_j \qquad (1 \leqslant k \leqslant m), \tag{5'}$$

with the initial conditions

$$(x_1, ..., x_m, z_1, ..., z_m)_{t=0} = (0, ..., 0, \alpha_1, ..., \alpha_m).$$

Let c_1, K satisfy $0 < c_1 < c$, $0 < K < \infty$. In the interval $| x_i | < c_1$, $| z_i | < K$ $(1 \leqslant i \leqslant m)$, the right-hand sides of the foregoing equations satisfy a Lipschitz condition.

From the existence and uniqueness theorem (see, e.g., Miller and Murray*, p. 42) for a system of ordinary differential equations we conclude:

There exists a constant $b_1 > 0$ and differentiable functions $x_i(t)$, $z_i(t)$ $(1 \leqslant i \leqslant m)$ in the interval $| t | \leqslant b_1$ such that

(i) $$\frac{dx_i(t)}{dt} = z_i(t) \qquad (1 \leqslant i \leqslant m), \qquad | t | < b_1,$$

$$\frac{dz_k(t)}{dt} = -\sum_{i,j=1}^{m} \Gamma_{ij}{}^k(x_1(t), ..., x_m(t))\, z_i(t)\, z_j(t) \qquad (1 \leqslant k \leqslant m), \qquad | t | < b_1;$$

(ii) $(x_1(t), ..., x_m(t), z_1(t), ..., z_m(t))_{t=0} = (0, ..., 0, \alpha_1, ..., \alpha_m)$;

(iii) $| x_i(t) | < c_1, | z_i(t) | < K \qquad (1 \leqslant i \leqslant m), \qquad | t | < b_1$;

(iv) $x_i(t), z_i(t) \qquad (1 \leqslant i \leqslant m)$ is the only set of functions satisfying the conditions (i), (ii), and (iii).

This shows that there exists a geodesic $t \to \gamma(t)$ in M satisfying (4) and that two such geodesics coincide in some interval around $t = 0$. Moreover, we can conclude from (iv) that if two geodesics $t \to \gamma_1(t)$ $(t \in I_1)$, $t \to \gamma_2(t)$ $(t \in I_2)$ coincide in some open interval, then they coincide for all $t \in I_1 \cap I_2$. Proposition 5.3 now follows immediately.

Definition. The geodesic with the properties in Prop. 5.3 will be denoted γ_X. If $X = 0$, we put $\gamma_X(t) = p$ for all $t \in \boldsymbol{R}$.

* K.S. Miller and F.J. Murray, Existence Theorems for Ordinary Differential Equations, New York Univ. Press, 1954

§ 6. The Exponential Mapping

Suppose again M is a C^∞ manifold with an affine connection. Let $p \in M$. We use the notation from the proof of Prop. 5.3. We shall now study the solutions of (5) and (5') and their dependence on the initial values. From the existence and uniqueness theorem (see, e.g., Miller and Murray p. 64) for the system (5), (5'), we can conclude:

There exists a constant b $(0 < b < c)$ and differentiable functions $\varphi_i(t, \xi_1, ..., \xi_m, \zeta_1, ..., \zeta_m)$ for $|t| \leqslant 2b$, $|\xi_i| \leqslant b$, $|\zeta_j| \leqslant b$ $(1 \leqslant i, j \leqslant m)$ such that:

(i) For each fixed set $(\xi_1, ..., \xi_m, \zeta_1, ..., \zeta_m)$ the functions

$$x_i(t) = \varphi_i(t, \xi_1, ..., \xi_m, \zeta_1, ..., \zeta_m)$$

$$z_i(t) = \left[\frac{\partial \varphi_i}{\partial t}\right](t, \xi_1, ..., \xi_m, \zeta_1, ..., \zeta_m), \qquad 1 \leqslant i \leqslant m, \qquad |t| \leqslant 2b,$$

satisfy (5) and (5') and $|x_i(t)| < c_1$, $|z_i(t)| < K$.

(ii) $(x_1(t), ..., x_m(t), z_1(t), ..., z_m(t))_{t=0} = (\xi_1, ..., \xi_m, \zeta_1, ..., \zeta_m)$.

(iii) The functions φ_i are uniquely determined by the above properties.

Theorem 6.1. *Let M be a manifold with an affine connection. Let p be any point in M. Then there exists an open neighborhood N_0 of 0 in M_p and an open neighborhood N_p of p in M such that the mapping $X \to \gamma_X(1)$ is a diffeomorphism of N_0 onto N_p.*

Proof. Using the notation above, we put

$$\psi_i(t, \zeta_1, ..., \zeta_m) = \varphi_i(t, 0, ..., 0, \zeta_1, ..., \zeta_m)$$

for $1 \leqslant i \leqslant m$, $|t| \leqslant 2b$, $|\zeta_i| \leqslant b$. Then

$$\psi_i(0, \zeta_1, ..., \zeta_m) = 0,$$

$$\left[\frac{\partial \psi_i}{\partial t}\right](0, \zeta_1, ..., \zeta_m) = \zeta_i.$$

Since $\gamma_X(st) = \gamma_{sX}(t)$, the uniqueness (iii) implies

$$\psi_i(st, \zeta_1, ..., \zeta_m) = \psi_i(t, s\zeta_1, ..., s\zeta_m) \tag{1}$$

The map $X \to \gamma_X(b)$ has coordinate expression

$$\Psi : (\zeta_1, \ldots, \zeta_m) \to (\psi_1(b, \zeta_1, \ldots, \zeta_m), \ldots \psi_m(b, \zeta_1, \ldots, \zeta_m))$$

and we calculate its Jacobian at $(0, \ldots, 0)$.

$$\left(\frac{\partial \psi_i}{\partial \zeta_j}\right)_{(0,\ldots,0)} = \lim_{h \to 0} \frac{\psi_i(b, 0, \ldots, hb, \ldots, 0) - \psi_i(b, 0, \ldots, 0)}{hb}.$$

Using (1) and the relation $0 = \psi_i(b, 0, \ldots, 0) = \psi_i(0, \ldots, b, \ldots, 0)$ this limit is

$$\lim_{h \to 0} \frac{\psi_i(hb, 0 \ldots, b, \ldots 0) - \psi_i(0, \ldots, b, \ldots 0)}{hb}$$
$$= \left(\frac{\partial \psi_i}{\partial t}\right) (0, \ldots, b, \ldots 0) = b\delta_{ij}.$$

Thus the Jacobian at $(0, \ldots 0)$ equals b^m. Since $\gamma_X(b) = \gamma_{bX}(1)$ the theorem follows.

Definition. The mapping $X \to \gamma_X(1)$ described in Theorem 6.1 is called the ***Exponential mapping*** at p and will be denoted by Exp (or Exp_p).

Definition. Let M be a manifold with an affine connection and p a point in M. An open neighborhood N_0 of the origin in M_p is said to be ***normal*** if: (1) the mapping Exp is a diffeomorphism of N_0 onto an open neighborhood N_p of p in M; (2) if $X \in N_0$, and $0 \leqslant t \leqslant 1$, then $tX \in N_0$.

The last condition means that N_0 is "***star-shaped.***" A neighborhood N_p of p in M is called a ***normal neighborhood*** of p if $N_p = \mathrm{Exp}\ N_0$ where N_0 is a normal neighborhood of 0 in M_p. Assuming this to be the case, and letting $X_1, \ldots, X_m$ denote some basis of M_p, the inverse mapping†

$$\mathrm{Exp}_p\,(a_1X_1 + \ldots + a_mX_m) \to (a_1, \ldots, a_m)$$

of N_p into R^m is called a system of ***normal coordinates*** at p.

§ 7. Covariant Differentiation

In § 5, parallelism was defined by means of the covariant differentiation ∇_X. Theorem 7.1 below shows that it is also possible to go the other way and describe the covariant derivative by means of parallel translation. This makes it possible to define the covariant derivative of other objects.

Definition. Let X be a vector field on a manifold M. A curve $s \to \varphi(s)$ ($s \in I$) is called an *integral curve* of X if

$$\dot{\varphi}(s) = X_{\varphi(s)}, \qquad s \in I. \tag{1}$$

Assuming $0 \in I$, let $p = \varphi(0)$ and let $\{x_1, ..., x_m\}$ be a system of coordinates valid in a neighborhood U of p. There exist functions

$X^i \in C^\infty(U)$ such that $X = \sum_i X^i \, \partial/\partial x_i$ on U. For simplicity let $x_i(s) = x_i(\varphi(s))$ and write X^i instead of $(X^i)^*$ (§ 2, No. 1). Then (1) is equivalent to

$$\frac{dx_i(s)}{ds} = X^i(x_1(s), ..., x_m(s)) \qquad (1 \leqslant i \leqslant m). \tag{2}$$

Therefore if $X_p \neq 0$ there exists an integral curve of X through p.

Theorem 7.1. *Let M be a manifold with an affine connection. Let $p \in M$ and let X, Y be two vector fields on M. Assume $X_p \neq 0$. Let $s \to \varphi(s)$ be an integral curve of X through $p = \varphi(0)$ and τ_t the parallel translation from p to $\varphi(t)$ with respect to the curve φ. Then*

$$(\nabla_X(Y))_p = \lim_{s \to 0} \frac{1}{s} (\tau_s^{-1} Y_{\varphi(s)} - Y_p).$$

Proof. We shall use the notation introduced above. Consider a fixed $s > 0$ and the family $Z_{\varphi(t)}$ ($0 \leqslant t \leqslant s$) which is parallel with respect to the curve φ such that $Z_{\varphi(0)} = \tau_s^{-1} Y_{\varphi(s)}$. We can write

$$Z_{\varphi(t)} = \sum_i Z^i(t) \left(\frac{\partial}{\partial x_i}\right)_{\varphi(t)}, \quad Y_{\varphi(t)} = \sum_i Y^i(t) \left(\frac{\partial}{\partial x_i}\right)_{\varphi(t)},$$

and have the relations

$$\dot{Z}^k(t) + \sum_{i,j} \Gamma_{ij}{}^k \dot{x}_i(t) Z^j(t) = 0 \qquad (0 \leqslant t \leqslant s)$$

$$Z^k(s) = Y^k(s) \qquad (1 \leqslant k \leqslant m).$$

By the mean value theorem

$$Z^k(s) = Z^k(0) + s\dot{Z}^k(t^*)$$

for a suitable number t^* between 0 and s. Hence the kth component of $(1/s)(\tau_s^{-1} Y_{\varphi(s)} - Y_p)$ is

$$\frac{1}{s}(Z^k(0) - Y^k(0)) = \frac{1}{s}\{Z^k(s) - s\dot{Z}^k(t^*) - Y^k(0)\}$$
$$= \sum_{i,j} \Gamma_{ij}{}^k(\varphi(t^*)) \dot{x}_i(t^*) Z^j(t^*) + \frac{1}{s}(Y^k(s) - Y^k(0)).$$

As $s \to 0$ this expression has the limit

$$\frac{dY^k}{ds} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{ds} Y^j.$$

Let this last expression be denoted by A_k. It was shown earlier that

$$\nabla_X(Y)_p = \sum_k A_k \left(\frac{\partial}{\partial x_k}\right)_p.$$

This proves the theorem.