2.11. **Main Theorem.**

Exercise 2.11.1. Show that for any $M \in \mathcal{M}$ the object $\underline{\mathrm{Hom}}(M, M)$ with the multiplication defined above is an algebra (in particular, define the unit morphism!).

Theorem 2.11.2. *Let $\mathcal{M}$ be a module category over $\mathcal{C}$, and assume that $M \in \mathcal{M}$ satisfies two conditions:*

1. The functor $\underline{Hom}(M, \bullet)$ is right exact (note that it is automatically left exact).

2. For any $N \in \mathcal{M}$ there exists $X \in \mathcal{C}$ and a surjection $X \otimes M \to N$.

Let $A = \underline{Hom}(M, M)$. Then the functor $F := \underline{Hom}(M, \bullet) : \mathcal{M} \to Mod_{\mathcal{C}}(A)$ is an equivalence of module categories.

Proof. We will proceed in steps:

(1) The map $F : \mathsf{Hom}(N_1, N_2) \to \mathsf{Hom}_A(F(N_1), F(N_2))$ is an isomorphism for any $N_2 \in \mathcal{M}$ and N_1 of the form $X \otimes M,\ X \in \mathcal{C}$.

Indeed, $F(N_1) = \underline{\mathrm{Hom}}(M, X \otimes M) = X \otimes A$ and the statement follows from the calculation:

$$\mathsf{Hom}_A(F(N_1), F(N_2)) = \mathsf{Hom}_A(X \otimes A, F(N_2)) = \mathsf{Hom}(X, F(N_2)) =$$

$$= \mathsf{Hom}(X, \underline{\mathrm{Hom}}(M, N_2)) = \mathsf{Hom}(X \otimes M, N_2) = \mathsf{Hom}(N_1, N_2).$$

(2) The map $F : \mathsf{Hom}(N_1, N_2) \to \mathsf{Hom}_A(F(N_1), F(N_2))$ is an isomorphism for any $N_1, N_2 \in \mathcal{M}$.

By condition 2, there exist objects $X, Y \in \mathcal{C}$ and an exact sequence

$$Y \otimes M \to X \otimes M \to N_1 \to 0.$$

Since F is exact, the sequence

$$F(Y \otimes M) \to F(X \otimes M) \to F(N_1) \to 0$$

is exact. Since Hom is left exact, the rows in the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathsf{Hom}(N_1, N_2) & \longrightarrow & \mathsf{Hom}(X \otimes M, N_2) & \longrightarrow & \mathsf{Hom}(Y \otimes M, N_2) \\ & \big\downarrow F & & \big\downarrow F & & \big\downarrow F \\ 0 \longrightarrow & \mathsf{Hom}(F(N_1), F(N_2)) & \longrightarrow & \mathsf{Hom}(F(X \otimes M), F(N_2)) & \longrightarrow & \mathsf{Hom}(F(Y \otimes M), F(N_2)) \end{array}$$

are exact. Since by step (1) the second and third vertical arrows are isomorphisms, so is the first one.

(3) The functor F is surjective on isomorphism classes of objects of $Mod_{\mathcal{C}}(A)$.

We know (see Exercise 2.9.15) that for any object $L \in Mod_{\mathcal{C}}(A)$ there exists an exact sequence

$$Y \otimes A \xrightarrow{\tilde{f}} X \otimes A \to L \to 0$$

for some $X, Y \in \mathcal{C}$. Let $f \in \mathsf{Hom}(Y \otimes M, X \otimes M)$ be the preimage of $\tilde{f}$ under the isomorphism

$$\mathsf{Hom}(Y \otimes M, X \otimes M) \cong \mathsf{Hom}_A(F(Y \otimes M), F(X \otimes M)) \cong \mathsf{Hom}_A(Y \otimes A, X \otimes A)$$

and let $N \in \mathcal{M}$ be the cokernel of f. It is clear that $F(N) = L$.

We proved that F is an equivalence of categories and proved the Theorem.

□

Remark 2.11.3. This Theorem is a special case of Barr-Beck Theorem in category theory, see [ML]. We leave it to the interested reader to deduce Theorem 2.11.2 from Barr-Beck Theorem.

We have two situations where condition 1 of Theorem 2.11.2 is satisfied:

1. $\mathcal{M}$ is an arbitrary module category over $\mathcal{C}$ and $M \in \mathcal{M}$ is projective.
2. $\mathcal{M}$ is an exact module category and $M \in \mathcal{M}$ is arbitrary.

Exercise 2.11.4. Check that in both of these cases $\underline{\mathrm{Hom}}(M, \bullet)$ is exact (Hint: in the first case first prove that $\underline{\mathrm{Hom}}(M, N)$ is a projective object of $\mathcal{C}$ for any $N \in \mathcal{M}$).

Exercise 2.11.5. Show that in both of these cases condition 2 is equivalent to the fact that $[M]$ generates $Gr(\mathcal{M})$ as $\mathbb{Z}_+$−module over $Gr(\mathcal{C})$.

Thus we have proved

Theorem 2.11.6. *(i) Let $\mathcal{M}$ be a finite module category over $\mathcal{C}$. Then there exists an algebra $A \in \mathcal{C}$ and a module equivalence $\mathcal{M} \simeq Mod_{\mathcal{C}}(A)$.*

(ii) Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$ and let $M \in \mathcal{M}$ be an object such that $[M]$ generates $Gr(\mathcal{M})$ as $\mathbb{Z}_+$−module over $Gr(\mathcal{C})$. Then there is a module equivalence $\mathcal{M} \simeq Mod_{\mathcal{C}}(A)$ where $A = \underline{Hom}(M, M)$.

2.12. **Categories of module functors.** Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two module categories over a multitensor category $\mathcal{C}$, and let (F, s), (G, t) be two module functors $\mathcal{M}_1 \to \mathcal{M}_2$.

Definition 2.12.1. A module functor morphism from (F, s) to (G, t) is a natural transformation a from F to G such that the following diagram commutes for any $X \in \mathcal{C}$, $M \in \mathcal{M}$:

$$\begin{array}{ccc} F(X \otimes M) & \xrightarrow{s} & X \otimes F(M) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle id \otimes a} \\ G(X \otimes M) & \xrightarrow{t} & X \otimes G(M) \end{array} \tag{2.12.1}$$

It is easy to see that the module functors with module functor morphisms introduced above form a category called the category of module functors. This category is very difficult to manage (consider the case $\mathcal{C} = \mathrm{Vec}$!) and we are going to consider a subcategory. Let $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ denote the full subcategory of the category of module functors consisting of *right exact* module functors (which are not necessarily left exact). First of all this category can be described in down to earth terms:

Proposition 2.12.2. *Assume that $\mathcal{M}_1 \simeq Mod_{\mathcal{C}}(A)$ and $\mathcal{M}_2 \simeq Mod_{\mathcal{C}}(B)$ for some algebras $A, B \in \mathcal{C}$. The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A-B-$bimodules via the functor which sends a bimodule M to the functor $\bullet \otimes_A M$.*

Proof. The proof repeats the standard proof from ring theory in the categorical setting. □

Thus we have the following

Corollary 2.12.3. *The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ of right exact module functors from $\mathcal{M}_1$ to $\mathcal{M}_2$ is abelian.*

Proof. Exercise. □

In a similar way one can show that the category of left exact module functors is abelian (using Hom over A instead of tensor product over A).

We would like now to construct new tensor categories in the following way: take a module category $\mathcal{M}$ and consider the category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ with composition of functors as a tensor product.

Exercise 2.12.4. The category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ has a natural structure of monoidal category.

But in general the category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is not rigid (consider the case $\mathcal{C} = \mathrm{Vec}$!). Thus to get a good theory (and examples of new tensor categories), we restrict ourselves to the case of exact module categories. We will see that in this case we can say much more about the categories $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ than in general.

2.13. **Module functors between exact module categories.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two exact module categories over $\mathcal{C}$. Note that the category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ coincides with the category of the additive module functors from $\mathcal{M}_1$ to $\mathcal{M}_2$ by Proposition 2.7.8.

Exercise 2.13.1. Any object of $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is of finite length.

Lemma 2.13.2. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be exact module categories over $\mathcal{C}$. The bifunctor of composition $Fun_{\mathcal{C}}(\mathcal{M}_2, \mathcal{M}_3) \times Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) \to Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_3)$ is biexact.*

Proof. This is an immediate consequence of Proposition 2.7.8. □

Another immediate consequence of Proposition 2.7.8 is the following:

Lemma 2.13.3. *Let $\mathcal{M}_1, \mathcal{M}_2$ be exact module categories over $\mathcal{C}$. Any functor $F \in Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ has both right and left adjoint.*

We also have the following immediate

Corollary 2.13.4. *Let $\mathcal{M}_1, \mathcal{M}_2$ be exact module categories over $\mathcal{C}$. Any functor $F \in Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ maps projective objects to projectives.*

In view of Example 2.6.6 this Corollary is a generalization of Theorem 1.49.3 (but this does not give a new proof of Theorem 1.49.3).

Proposition 2.13.5. *The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is finite.*

Proof. We are going to use Theorem 2.11.2. Thus $\mathcal{M}_1 = Mod_{\mathcal{C}}(A_1)$ and $\mathcal{M}_2 = Mod_{\mathcal{C}}(A_2)$ for some algebras $A_1, A_2 \in \mathcal{C}$. It is easy to see that the category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $(A_1, A_2)-$bimodules. But this category clearly has enough projective objects: for any projective $P \in \mathcal{C}$ the bimodule $A_1 \otimes P \otimes A_2$ is projective. □

2.14. **Dual categories.** Observe that the adjoint to a module functor has itself a natural structure of a module functor (we leave it to the reader to define this). In particular, it follows that the category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a rigid monoidal category.

Definition 2.14.1. We denote this category as $\mathcal{C}^*_{\mathcal{M}}$ and call it the *dual category* to $\mathcal{C}$ with respect to $\mathcal{M}$.

By Proposition 2.13.5, this category is finite.

Remark 2.14.2. This notion is a categorical version of notion of the endomorphism ring of a module (i.e., a centralizer algebra), and gives many new examples of tensor categories.

Lemma 2.14.3. *The unit object $\mathbf{1} \in \mathcal{C}^*_{\mathcal{M}}$ is a direct sum of projectors to subcategories $\mathcal{M}_i$. Each such projector is a simple object.*

Proof. The first statement is clear. For the second statement it is enough to consider the case when $\mathcal{M}$ is indecomposable. Let F be a nonzero module subfunctor of the identity functor. Then $F(X) \neq 0$ for any $X \neq 0$. Hence $F(X) = X$ for any simple $X \in \mathcal{M}$ and thus $F(X) = X$ for any $X \in \mathcal{M}$ since F is exact. □

Thus, the category $\mathcal{C}_{\mathcal{M}}^*$ is a finite multitensor category; in particular if $\mathcal{M}$ is indecomposable then $\mathcal{C}_{\mathcal{M}}^*$ is finite tensor category. Note that by the definition $\mathcal{M}$ is a module category over $\mathcal{C}_{\mathcal{M}}^*$.

Lemma 2.14.4. *The module category $\mathcal{M}$ over $\mathcal{C}_{\mathcal{M}}^*$ is exact.*

Proof. Let $A \in \mathcal{C}$ be an algebra such that $\mathcal{M} = Mod_{\mathcal{C}}(A)$. Thus the category $\mathcal{C}_{\mathcal{M}}^*$ is identified with the category $\mathsf{Bimod}(A)^{op}$ of $A-$bimodules with opposite tensor product (because $A-$bimodules act naturally on $Mod_{\mathcal{C}}(A)$ from the right). Any projective object in the category of $A-$bimodules is a direct summand of the object of the form $A \otimes P \otimes A$ for some projective $P \in \mathcal{C}$. Now for any $M \in Mod_{\mathcal{C}}(A)$ one has that $M \otimes_A A \otimes P \otimes A = (M \otimes P) \otimes A$ is projective by exactness of the category $Mod_{\mathcal{C}}(A)$. The Lemma is proved. $\square$

Example 2.14.5. It is instructive to consider the internal Hom for the category $Mod_{\mathcal{C}}(A)$ considered as a module category over $\mathcal{C}_{\mathcal{M}}^* = \mathsf{Bimod}(A)$. We leave to the reader to check that $\underline{\mathrm{Hom}}_{\mathcal{C}_{\mathcal{M}}^*}(M, N) = {}^*M \otimes N$ (the right hand side has an obvious structure of $A-$bimodule). In particular $B = \underline{\mathrm{Hom}}_{\mathcal{C}_{\mathcal{M}}^*}(A, A) = {}^*A \otimes A$ is an algebra in the category of $A-$bimodules. Thus B is an algebra in the category $\mathcal{C}$ and it is easy to see from definitions that the algebra structure on $B = {}^*A \otimes A$ comes from the evaluation morphism $ev : A \otimes {}^*A \to \mathbf{1}$. Moreover, the coevaluation morphism induces an embedding of algebras $A \to {}^*A \otimes A \otimes A \to {}^*A \otimes A = B$ and the $A-$bimodule structure of B comes from the left and right multiplication by A.

Thus for any exact module category $\mathcal{M}$ over $\mathcal{C}$ the category $(\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$ is well defined. There is an obvious tensor functor $can : \mathcal{C} \to (\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$.

Theorem 2.14.6. *The functor $can : \mathcal{C} \to (\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$ is an equivalence of categories.*

Proof. Let A be an algebra such that $\mathcal{M} = Mod_{\mathcal{C}}(A)$. The category $\mathcal{C}_{\mathcal{M}}^*$ is identified with the category $\mathsf{Bimod}(A)^{op}$. The category $(\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$ is identified with the category of $B-$bimodules in the category of $A-$bimodules (here B is the same as in Example 2.14.5 and is considered as an algebra in the category of $A-$modules). But this latter category is tautologically identified with the category of $B-$bimodules (here B is an algebra in the category $\mathcal{C}$) since for any $B-$module one reconstructs the $A-$module structure via the embedding $A \to B$ from Example 2.14.5. We are going to use the following

Lemma 2.14.7. *Any left $B-$module is of the form ${}^*A \otimes X$ for some $X \in \mathcal{C}$ with the obvious structure of an $A-$module. Similarly, any right $B-$module is of the form $X \otimes A$.*

Proof. Let us consider $\mathcal{C}$ as a module category over itself. Consider an object ${}^*A \in \mathcal{C}$ as an object of this module category. Then by Example 2.10.8 $\underline{\mathrm{Hom}}({}^*A, {}^*A) = {}^*A \otimes A = B$ and the statement follows from Theorem 2.11.2. The case of right modules is completely parallel. $\square$

It follows from the Lemma that any $B-$bimodule is of the form ${}^*A \otimes X \otimes A$ and it is easy to see that $can(X) = {}^*A \otimes X \otimes A$. The Theorem is proved. $\square$

Remark 2.14.8. Theorem 2.14.6 categorifies the classical "double centralizer theorem" for projective modules, which says that if A is a finite dimensional algebra and P is a projective A-module then the centralizer of $\mathrm{End}_A(P)$ in P is A.

Corollary 2.14.9. *Assume that $\mathcal{C}$ is a finite tensor (not only multitensor) category. Then an exact module category $\mathcal{M}$ over $\mathcal{C}$ is indecomposable over $\mathcal{C}^*_{\mathcal{M}}$.*

Proof. This is an immediate consequence of Theorem 2.14.6 and Lemma 2.14.3. $\square$

Let $\mathcal{M}$ be a fixed module category over $\mathcal{C}$. For any other module category $\mathcal{M}_1$ over $\mathcal{C}$ the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ has an obvious structure of a module category over $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$.

Lemma 2.14.10. *The module category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ over $\mathcal{C}^*_{\mathcal{M}}$ is exact.*

Proof. Assume that $\mathcal{M} = Mod_{\mathcal{C}}(A)$ and $\mathcal{M}_1 = Mod_{\mathcal{C}}(A_1)$. Identify $\mathcal{C}^*_{\mathcal{M}}$ with the category of $A-$bimodules and $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ with the category of $(A_1 - A)-$bimodules. Any projective object of $\mathsf{Bimod}(A)$ is a direct summand of an object of the form $A \otimes P \otimes A$ for some projective $P \in \mathcal{C}$. Let M be an $(A_1 - A)-$bimodule, then $M \otimes_A A \otimes P \otimes A = M \otimes P \otimes A$. Now $\mathsf{Hom}_{A_1 - A}(M \otimes P \otimes A, \bullet) = \mathsf{Hom}_{A_1}(M \otimes P, \bullet)$ (here $\mathsf{Hom}_{A_1 - A}$ is the Hom in the category of $(A_1 - A)-$bimodules and Hom_{A_1} is the Hom in the category of left A_1-modules) and it is enough to check that $M \otimes P$ is a projective left A_1-module. This is equivalent to $(M \otimes P)^*$ being injective (since $N \mapsto N^*$ is an equivalence of the category of left $A-$modules to the category of right $A-$modules). But $(M \otimes P)^* = P^* \otimes M^*$ and results follows from projectivity of P^* and Lemma 2.7.3. $\square$

The proof of the following Theorem is similar to the proof of Theorem 2.14.6 and is left to the reader.

Theorem 2.14.11. *Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. The maps $\mathcal{M}_1 \mapsto Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ and $\mathcal{M}_2 \mapsto Fun_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}_2, \mathcal{M})$ are mutually inverse bijections of the sets of equivalence classes of exact module categories over $\mathcal{C}$ and over $\mathcal{C}^*_{\mathcal{M}}$.*

Following Müger, [Mu], we will say that the categories $\mathcal{C}$ and $(\mathcal{C}^*_{\mathcal{M}})^{op}$ are *weakly Morita equivalent.*

Example 2.14.12. Let $\mathcal{C}$ be a finite multitensor category. Then $\mathcal{C}$ is an exact module category over $\mathcal{C} \boxtimes \mathcal{C}^{op}$.

Definition 2.14.13. The corresponding dual category $\mathcal{Z}(\mathcal{C}) := \mathcal{C}^*_{\mathcal{C}\boxtimes\mathcal{C}^{op}}$ is called the *Drinfeld center* of $\mathcal{C}$.

This notion categorifies the notion of the center of a ring, since the center of a ring A is the ring of endomorphisms of A as an A-bimodule.

Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. For $X, Y \in \mathcal{M}$ we have two notions of internal Hom — with values in $\mathcal{C}$ and with values in $\mathcal{C}^*_{\mathcal{M}}$, denoted by $\underline{\mathrm{Hom}}_{\mathcal{C}}$ and $\underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}$ respectively. The following simple consequence of calculations in Examples 2.10.8 and 2.14.5 is very useful.

Proposition 2.14.14. ("Basic identity") *Let $X, Y, Z \in \mathcal{M}$. There is a canonical isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \otimes Z \simeq {}^*\underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(Z, X) \otimes Y.$$

Proof. By Theorem 2.14.6 it is enough to find a canonical isomorphism

$${}^*\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, X) \otimes Y \simeq \underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(X, Y) \otimes Z.$$

This isomorphism is constructed as follows. Choose an algebra A such that $\mathcal{M} = Mod_{\mathcal{C}}(A)$. By Example 2.10.8 the LHS is ${}^*(X \otimes^A Z^*) \otimes Y = {}^*(Z \otimes_A {}^*X)^* \otimes Y = (Z \otimes_A {}^*X) \otimes Y$. On the other hand by Example 2.14.5 the RHS is $Z \otimes_A ({}^*X \otimes Y)$. Thus the associativity isomorphism gives a canonical isomorphism of the LHS and RHS. Observe that the isomorphism inverse to the one we constructed is the image of the identity under the homomorphism

$$\mathsf{Hom}(Y, Y) \to \mathsf{Hom}(\underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(X, Y) \otimes X, Y) \to$$

$$\mathsf{Hom}(\underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(X, Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, X) \otimes Z, Y) \simeq$$

$$\mathsf{Hom}(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, X) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(X, Y) \otimes Z, Y) \simeq$$

$$\mathsf{Hom}(\underline{\mathrm{Hom}}_{\mathcal{C}^*_{\mathcal{M}}}(X, Y) \otimes Z, {}^*\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, X) \otimes Y)$$

and thus does not depend on the choice of A. $\square$

Remark 2.14.15. Following [Mu] one can construct from $\mathcal{M}$ a 2-category with 2 objects $\mathfrak{A}, \mathfrak{B}$ such that $\mathsf{End}(\mathfrak{A}) \cong \mathcal{C}$, $\mathsf{End}(\mathfrak{B}) \cong (\mathcal{C}^*_{\mathcal{M}})^{op}$, $\mathsf{Hom}(\mathfrak{A}, \mathfrak{B}) \cong \mathcal{M}$, and $\mathsf{Hom}(\mathfrak{B}, \mathfrak{A}) = Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{C})$. In this language Proposition 2.14.14 expresses the associativity of the composition of Hom's.

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