# 18.781 Problem Set 8

Thursday, May 3.
Collaboration is allowed and encouraged. However, your writeups should be your own, and you must note on the front the names of the students you worked with.
Extensions will only be given for extenuating circumstances.

1. Let $n$ be a positive integer. Evaluate
$$\sum_{k=0}^{n} \binom{n}{3k}.$$
[Hint: use the generating function $(1+x)^n = \sum \binom{n}{k} x^k$]

2. For a sequence $\{a_n\}$, we can define another kind of generating function, called an exponential generating function, as follows:
$$\tilde{A}(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}.$$
It satisfies many of the nice properties we have seen for generating functions (for example, linearity with respect to the sequence). But some properties are slightly modified.

   (a) Show that the exponential generating function for the left-shifted sequence $\{a_1, a_2, \ldots, \}$ is $\frac{d}{dx}\tilde{A}(x)$.

   (b) If $\tilde{A}(x)$ is the exponential generating function for $\{a_n\}$ and $\tilde{B}(x)$ is the exponential generating function for $\{b_n\}$, show that $\tilde{A}(x)\tilde{B}(x)$ is the exponential generating function for the sequence $\{c_n\}$ given by
$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

   (c) Show that the exponential generating function $E(x)$ for the sequence $a_n = r^n$ (where $r$ is some fixed complex number) satisfies $E'(x) = rE(x)$. Solve this differential equation to deduce that
$$e^{rx} = \sum_{n \geq 0} \frac{r^n x^n}{n!}.$$

3. Define a sequence $B_n$ by the identity
$$f(x) := \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$
i.e. $B_n$ is $n!$ times the coefficient of $x^n$ in the expansion of the left hand side, where one uses $e^x = \sum_{n=0}^{\infty} x^n/n!$.

   (a) Calculate $B_0$ through $B_{10}$ (you may use gp).

(b) Show that for $n > 1$ odd, $B_n = 0$. [Hint: consider $f(x) - f(-x)$ ]

(c) Establish the recurrence (for $n \geq 2$)

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0.$$

[Hint: multiply both sides of the defining equation by $e^x - 1$]

(d) Let $S_k(n) = 1^k + 2^k + \cdots + n^k$ be the sum of the $k$'th powers of the first $n$ natural numbers. Show that the $S_k$ is given by the following *polynomial* of degree $k+1$ in $n$:

$$S_k(n) = \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} B_i n^{k+1-i}.$$

[Hint: calculate $\sum_{k \geq 0} S_k(n) x^k / k!$]

4. (a) Let $m$ and $n$ be two integers which are sums of two squares. Show that $mn$ is a sum of two squares. Use this to show that any positive integer of the form $\prod p_i^{e_i} \prod q_j^{f_j}$, where $p_i$ are primes which are 2 or 1 mod 4, and $q_i$ are primes which are 3 mod 4, *such that* $f_j$ are al even, is a sum of two integer squares.

   (b) Now suppose $n$ is a sum of two integer squares. Show that it must have the form above, i.e. if a prime $q$ which is 3 mod 4 divides $n$, then it must divide it to an even power. [Hint: If $n = a^2 + b^2$, show $q$ must divide $a$ and $b$. Induct.]

   (c) Show that $n$ is a sum of squares of two rational numbers if and only if it's a sum of squares of two integers.

5. (a) Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. Write down the cyclotomic polynomial $\Phi_3(x)$ and thereby compute $\omega^2$ in terms of $\omega$. Now calculate the norm of the complex number $a + b\omega$. Use this to show that if $m, n$ are two integers which can be written in the form $a^2 - ab + b^2$, then their product can also be written likewise.

   (b) Show that if $p$ is a prime which can be written as $a^2 - ab + b^2$, then $p$ cannot be 2 (mod 3), i.e. $p = 3$ or $p \equiv 1 \pmod 3$.

6. (a) Show 3, and any prime $p$ which is $1 \pmod 3$, can be written as a $a^2 - ab + b^2$, for some integers $a$ and $b$. [Hint: imitate the proof from class for the sum of two squares]

   (b) Show that an integers $n$ can be written as $a^2 - ab + b^2$ for integers $m, n$ if and only if $n$ is positive and every prime which is 2 mod 3 and divides $n$, divides it to an even power.

7. Calculate the continued fractions of 6157/783 and $\sqrt{15}$.

8. (Bonus)

   (a) Take logs and differentiate the relation

$$\sin(z) = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

to show that

$$z \cot z = 1 + 2 \sum_{n \geq 1} \frac{z^2}{z^2 - n^2 \pi^2} = 1 - 2 \sum_{n \geq 1} \sum_{k \geq 1} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}}.$$

In $x/(e^x - 1) = \quad B_r x^r/r!$, plug in $x = 2iz$ adnd use $e^{iz} = \cos(z) + i \sin(z)$ to deduce (recalling that $B_{2k+1} = 0$ for $k \geq 1$) that

$$z \cot z = 1 - \sum_{k \geq 1} (-1)^{k-1} B_{2k} \frac{2^{2k} z^{2k}}{(2k)!}.$$

Conclude that

$$\zeta(2k) = (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}.$$

18.781 Theory of Numbers
Spring 2012