

LECTURE 12

Derived Functors and Explicit Projective Resolutions

Let X and Y be complexes of A -modules. Recall that in the last lecture we defined $\underline{\mathrm{Hom}}_A(X, Y)$, as well as $\underline{\mathrm{Hom}}_A^{\mathrm{der}}(X, Y) := \underline{\mathrm{Hom}}_A(P, Y)$ for a projective complex $P \xrightarrow{\mathrm{qis}} X$, i.e., a *projective resolution* of X . We also defined the *Ext-groups* $\mathrm{Ext}_A^i(X, Y) := H^i \underline{\mathrm{Hom}}_A^{\mathrm{der}}(X, Y)$. The most important example in this case is $A := \mathbb{Z}[G]$, where $X^{\mathrm{h}G} := \underline{\mathrm{Hom}}_A^{\mathrm{der}}(\mathbb{Z}, X)$ are the *homotopy invariants* of X . This construction has the basic properties that $\underline{\mathrm{Hom}}_A^{\mathrm{der}}(X, -)$ preserves quasi-isomorphisms, and $P \xrightarrow{\mathrm{qis}} X$ is unique up to homotopy, and such homotopies are unique up to homotopy, which are unique up to homotopy, and so on.

As an aside, note that we can actually define the derived functor $\underline{\mathrm{Hom}}^{\mathrm{der}}(X, -)$ more canonically, without choosing a particular projective resolution, via

$$\underline{\mathrm{Hom}}^{\mathrm{der}}(X, Y) := \varinjlim_{\substack{P \xrightarrow{\mathrm{qis}} X \\ \text{projective}}} \underline{\mathrm{Hom}}(P, Y),$$

where the P are ordered by maps of chain complexes

$$P' \longrightarrow P \xrightarrow{\mathrm{qis}} X, \\ \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad \mathrm{qis}$$

which forcibly removes the choice of P .

CLAIM 12.1. *Suppose we have a map of chain complexes $f: X_1 \rightarrow X_2$, which have projective resolutions P_1 and P_2 , respectively. Then we have a map $\varphi: P_1 \rightarrow P_2$ such that the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \mathrm{qis} \uparrow & & \mathrm{qis} \uparrow \\ P_1 & \xrightarrow{\varphi} & P_2 \end{array}$$

Moreover, such a map is unique up to homotopy.

PROOF. Because the derived functor preserves quasi-isomorphisms, the induced map of complexes of maps

$$\underline{\mathrm{Hom}}(P_1, P_2) \xrightarrow{\mathrm{qis}} \underline{\mathrm{Hom}}(P_1, X_2)$$

is a quasi-isomorphism. We are given a map, namely the composition $P_1 \rightarrow X_1 \rightarrow X_2$, which is killed by the differential since it is a map of chain complexes, and therefore defines a cohomology class in $H^0 \underline{\mathrm{Hom}}(P_1, X_2)$. So there is some cohomology class in $H^0 \underline{\mathrm{Hom}}(P_1, P_2)$ which is a lift of that map through P_2 , which is well-defined and unique up to homotopy. \square

The upshot is that, for every chain complex Y , we get a map

$$\underline{\mathrm{Hom}}_A^{\mathrm{der}}(X_2, Y) \rightarrow \underline{\mathrm{Hom}}_A^{\mathrm{der}}(X_1, Y)$$

by pulling back along φ . A quick “application” is the following:

CLAIM 12.2. *If $H \subseteq G$ is a subgroup and X is a complex of G -modules, then we get a restriction map $X^{\mathrm{h}G} \rightarrow X^{\mathrm{h}H}$ (which is well-defined up to homotopy).*

Intuitively, something which is G -invariant is also H -invariant.

PROOF. Consider

$$\mathbb{Z}[G/H] = \{f: G/H \rightarrow \mathbb{Z} \mid f \text{ nonzero at finitely many points}\},$$

which has a G -action via translations and is equivalent to the induced module from H to G by the trivial module, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z}$.

CLAIM 12.3. $\underline{\mathrm{Hom}}_G^{\mathrm{der}}(\mathbb{Z}[G/H], X) \simeq X^{\mathrm{h}H}$ is a quasi-isomorphism.

PROOF. Let $P_H \xrightarrow{\mathrm{qis}} \mathbb{Z}$, where P_H is a projective complex of H -modules. Then we have an induced G -module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P_H$. Note that $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$ module, as choosing coset representatives for G/H yields a basis. Therefore, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} -$ preserves quasi-isomorphisms (we know this for $\mathbb{Z}[H]$, and then we may regard $\mathbb{Z}[G]$ as a direct sum of copies of $\mathbb{Z}[H]$). This implies that

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P_H \xrightarrow{\mathrm{qis}} \mathbb{Z}[G/H],$$

which is projective as a complex of $\mathbb{Z}[G]$ -modules. This is because both $\mathbb{Z}[G]$ and P_H are bounded, so it will be bounded, and inducing up to $\mathbb{Z}[G]$ preserves projective modules as we will still obtain a direct summand of a free module. Alternatively, we could use the universal property that every map to an acyclic complex is null-homotopic, as a G -equivariant map out of the induced complex is the same as an H -equivariant map out of P_H . This gives the claim, as

$$\underline{\mathrm{Hom}}_G^{\mathrm{der}}(\mathbb{Z}[G/H], X) := \underline{\mathrm{Hom}}_G(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P_H, X) = \underline{\mathrm{Hom}}_H(P_H, X) =: X^{\mathrm{h}H},$$

by definition. \square

The upshot is that we get a map $X^{\mathrm{h}G} \rightarrow X^{\mathrm{h}H}$ via

$$\begin{aligned} \epsilon: \mathbb{Z}[G/H] &\rightarrow \mathbb{Z} \\ \sum_{g_i \in G/H} n_i g_i &\mapsto \sum_i n_i, \end{aligned}$$

which is clearly a G -equivariant map when we equip \mathbb{Z} with the trivial action. By the previous discussion, we have a restriction map of derived functors

$$X^{\mathrm{h}G} = \underline{\mathrm{Hom}}_G^{\mathrm{der}}(\mathbb{Z}, X) \rightarrow \underline{\mathrm{Hom}}_G^{\mathrm{der}}(\mathbb{Z}[G/H], X) = X^{\mathrm{h}H},$$

which is well-defined up to homotopy (defined up to homotopy, etc., our “usual error”). \square

Recall that everything here is a complex of abelian groups, so there is no “type incompatibility”. In fact, if $H \leq G$ is finite index, then we have a G -equivariant map

$$\mathbb{Z} \xrightarrow{\kappa} \mathbb{Z}[G/H] \xrightarrow{\epsilon} \mathbb{Z}$$

$$1 \mapsto \sum_{g \in G/H} g,$$

such that the composition corresponds to multiplication by the index $[G : H]$. This gives an *inflation map* $X^{\text{h}H} \rightarrow X^{\text{h}G}$ such that the composition $X^{\text{h}G} \rightarrow X^{\text{h}H} \rightarrow X^{\text{h}G}$ is homotopic to multiplication by $[G : H]$.

More concretely, suppose we had an H -invariant object and a G -invariant object. Taking coset representatives of G/H , we could take the “relative norm” of any H -invariant element, which would yield a G -invariant element. This is precisely what our maps are doing above, and explains why the composition multiplies by $[G : H]$.

DEFINITION 12.4. $H^i(G, X) := H^i(X^{\text{h}G})$ is the (*hyper-*)*cohomology* of G with coefficients in X .

The prefix “hyper” used to refer to an operation on complexes; if the complex was only in degree 0, it would be called “group cohomology.”

CLAIM 12.5. *If X is only in non-negative degrees, that is, $X^i = 0$ for all $i < 0$, then $H^0(G, X) = H^0(X)^G$ and $H^i(G, X) = 0$ for $i < 0$.*

PROOF. Choose some projective resolution P of \mathbb{Z} as a G -module, which by the construction in Proposition 11.9 can be taken to be in non-positive degrees only. By definition, $H^0(G, X)$ is equivalent to the homotopy classes of maps $f: P \rightarrow X$, all of which look like

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \xrightarrow{d} & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow f & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X^0 & \xrightarrow{d} & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \end{array}$$

Thus, any homotopy of f is 0, and $H^i(G, X) = 0$ for $i < 0$ similarly. By commutativity, we must have $df = fd = 0$. It follows that such maps f are equivalent to G -equivariant maps

$$\mathbb{Z} = P^0/dP^{-1} = \text{Coker}(P^{-1} \rightarrow P^0) \rightarrow \text{Ker}(X^0 \rightarrow X^1) = H^0(X)$$

by quasi-isomorphism, which is equivalent to a G -invariant vector in the cohomology $H^0(X)$ (i.e., via the image of 1). \square

We now turn to the problem of constructing explicit projective resolutions of \mathbb{Z} as a G -Module.

EXAMPLE 12.6. Let $G := \mathbb{Z}/n\mathbb{Z}$ with generator σ . We claim that the following is a quasi-isomorphism:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\sum_i \sigma^i} & \mathbb{Z}[G] & \xrightarrow{1-\sigma} & \mathbb{Z}[G] & \xrightarrow{\sum_i \sigma^i} & \mathbb{Z}[G] & \xrightarrow{1-\sigma} & \mathbb{Z}[G] & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \epsilon & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

The vanishing of the negative cohomologies follows from our earlier results on Tate cohomology, and the kernel of ϵ , i.e., elements whose coordinates sum to zero, is the image of $1 - \sigma$.

COROLLARY 12.7. *If M is a G -module (thought of as a complex in degree 0, then $M^{\text{h}G}$ is quasi-isomorphic to the complex*

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow M \xrightarrow{1-\sigma} M \xrightarrow{\sum_i \sigma^i} M \xrightarrow{1-\sigma} \cdots,$$

where the first M is in degree 0.

Note that a G -equivariant map from $\mathbb{Z}[G]$ to any object is that object. Indeed, the invariants are the zeroth cohomology group, as desired. Thus, this construction gives “half of” what we learned earlier with Tate cohomology.

Now we’d like to construct an explicit resolution for every G . Throughout, our “motto” will be that “all such things come from the bar construction.” Let A be a commutative ring, and B an A -algebra; the most important case will be $A := \mathbb{Z}$ and $B := \mathbb{Z}[G]$.

DEFINITION 12.8. For all such A and B , the *bar complex* $\text{Bar}_A(B)$ is

$$\cdots \rightarrow B \otimes_A B \otimes_A B \xrightarrow{b_1 \otimes b_2 \otimes b_2 \mapsto b_1 b_2 \otimes b_3 - b_1 \otimes b_2 b_3} B \otimes_A B \xrightarrow{b_1 \otimes b_2 \mapsto b_1 b_2} B \rightarrow 0 \rightarrow \cdots$$

with B in degree 0. In general, $\text{Bar}_A^{-n}(B) := B^{\otimes_A n+1}$, with differential

$$\begin{aligned} b_1 \otimes \cdots \otimes b_{n+1} \mapsto & b_1 b_2 \otimes b_3 \otimes \cdots \otimes b_{n+1} \\ & - b_1 \otimes b_2 b_3 \otimes \cdots \otimes b_{n+1} \\ & + b_1 \otimes b_2 \otimes b_3 b_4 \otimes \cdots \otimes b_{n+1} \\ & - \cdots. \end{aligned}$$

It’s easy enough to see that this differential squares to zero by selectively removing tensors and checking signs, so this is indeed a chain complex.

CLAIM 12.9. $\text{Bar}_A(B)$ is homotopy equivalent to zero.

PROOF. We’d like a null-homotopy of the identity map of $\text{Bar}_A(B)$, that is, a map h such that $hd + dh + \text{id}$:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & B \otimes_A B \otimes_A B & \longrightarrow & B \otimes_A B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \\ & \swarrow h^2 & \downarrow \text{id} & \swarrow h^1 & \downarrow \text{id} & \swarrow h^0 & \downarrow \text{id} & \swarrow & \downarrow & & \\ \cdots & \longrightarrow & B \otimes_A B \otimes_A B & \longrightarrow & B \otimes_A B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

So define $h^0(b) := 1 \otimes b$, and $h^1(b_1 \otimes b_2) := 1 \otimes b_1 \otimes b_2$. Indeed, we then have

$$(dh' + h^0 d)(b_1 \otimes b_2) = d(1 \otimes b_1 \otimes b_2) + 1 \otimes b_1 b_2 = b_1 \otimes b_1 - 1 \otimes b_1 b_2 + 1 \otimes b_1 b_2 = b_1 \otimes b_2,$$

as desired. It’s easy to show that defining h^n similarly for all n gives a null-homotopy of the identity. \square

As a reformulation, consider the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & B \otimes_A B \otimes_A B & \longrightarrow & B \otimes_A B & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow d & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 \longrightarrow \cdots, \end{array}$$

where d is the multiplication map in the differential. This is a homotopy equivalence, since its cone is $\text{Bar}_A(B)$.

Consider each term as a bimodule (that is, a module with commuting actions on the left and right), where we multiply in the first term by B on the left, and

multiply in the last term by B on the right. These differentials are then bimodule homomorphisms. Then given a (left) B -module M , we can tensor over B with M , which yields a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & B \otimes_A B \otimes_A M & \longrightarrow & B \otimes_A M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow^{b \otimes m \mapsto bm} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

that is also a homotopy equivalence (the map $B \otimes_A M \rightarrow M$ is the “action map”; also note that these tensor products make sense since B is an A -module). The differentials are the same, except the last term is replaced with an element of m , so for instance we have

$$b_1 \otimes b_2 \otimes m \mapsto b_1 b_2 \otimes m - b_1 \otimes b_2 m.$$

In words, M is canonically homotopy equivalent to a complex where every term is of the form $B \otimes_A N$, where in this case N stands for $B \otimes_A \cdots \otimes_A B \otimes_A M$, that is, a module induced from some A -module.



We now apply this to the case where $A := \mathbb{Z}$, $B := \mathbb{Z}[G]$, and $M := \mathbb{Z}$, i.e., the trivial module. Note that this is only a quasi-isomorphism of complexes of B -modules, and not a homotopy equivalence, as the inverse is only A -linear, and not B -linear! Indeed, note that such an inverse would be

$$\begin{array}{ccc} B \otimes_A M & \xrightarrow{b \otimes m \mapsto bm} & M \xrightarrow{m \mapsto 1 \otimes m} B \otimes_A M \\ b \otimes m & \mapsto bm & \mapsto 1 \otimes bm \neq b \otimes m = b(1 \otimes m), \end{array}$$

since the action is on B , not M , and is not an action of B -modules. In general, existence of a quasi-isomorphism in one direction does not imply existence of one in the other direction, whereas by fiat homotopy equivalence includes a map in the other direction and is therefore reflexive. Also, recall that homotopy equivalence implies quasi-isomorphism.

Thus, we obtain a canonical projective resolution of \mathbb{Z} by free G -modules

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z}[G^3] & \longrightarrow & \mathbb{Z}[G \times G] & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow^\epsilon & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots, \end{array}$$

with differentials $(g_1, g_2) \mapsto g_1 g_2 - g_1$, and so forth, since G acts trivially on \mathbb{Z} . Note that $\mathbb{Z}[G \times G] \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, since both have a basis by the elements of the product group.

This is a great explicit projective resolution of \mathbb{Z} for computing group cohomology! We end up with a complex of the form

$$\cdots \rightarrow 0 \rightarrow M \rightarrow \mathbb{Z}[G] \otimes M \rightarrow \mathbb{Z}[G \times G] \otimes M \rightarrow \cdots,$$

with M in degree 0 and G finite. Elements in $\mathbb{Z}[G] \otimes M$ in the kernel of the differential are called *group n -cocycles* with coefficients in M ; elements in the image of the differential are called *n -coboundaries*.

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Spring 2016

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