The separation axioms

We give two examples of spaces that satisfy a given separation axiom but not the next stronger one. The first is a familiar space, and the second is not.

Theorem F.1. If J is uncountable, $\mathbb{R}^J$ is completely regular, but not normal.

Proof. The proof follows the outline given in Exercise 9 of §32. The space $\mathbb{R}^J$ is of course completely regular, being a product of completely regular spaces. Let $X = \mathbb{Z}_+^J$; since X is a closed subspace of $\mathbb{R}^J$, it suffices to show that X is not normal. We shall use functional notation for the elements of X rather than tuple notation.

Given a finite subset B of J, and given a point x of X, let $U(x,B)$ be the set of all those elements y of X such that $y(\alpha) = x(\alpha)$ for all α in B. Then $U(x,B)$ is open in X; indeed, it is the cartesian product $\prod U_\alpha$, where U_α is a one-point set for α in B and $U_\alpha = \mathbb{Z}_+$ otherwise. It is immediate that the sets $U(x,B)$ form a basis for X, since the one-point sets form a basis for $\mathbb{Z}_+$.

Given a positive integer n, let P_n be the subset of X consisting of those maps $x : J \to \mathbb{Z}_+$ such that for each i different from n, the set $x^{-1}(i)$ consists of at most one element of J. (This of course implies that $x^{-1}(n)$ consists of uncountably many elements of J.) The set P_n is closed, for if y is not in P_n, then there is an integer $i \neq n$ and distinct indices α, β of J such that $y(\alpha) = y(\beta) = i$. The basis element $U(y,B)$, where $B = \{\alpha, \beta\}$, contains y and is disjoint from P_n.

Furthermore, if $n \neq m$, then P_n and P_m are disjoint. For if x is in P_n, then x maps uncountably many elements of J to n; while if x is in P_m, it maps at most one element of J to n.

Let U and V be open sets of X containing P_1 and P_2, respectively. We show that U and V are not disjoint. It follows that X is not normal.

Step 1. We define a sequence $\alpha_1, \alpha_2, \ldots$ of elements of J, a sequence $x_1, x_2, \ldots$ of points of X, and a sequence $n_1 < n_2 < \ldots$ of positive integers, inductively as follows:

Let $x_1(\alpha) = 1$ for all α. Then x_1 is in P_1; choose a finite nonempty subset B_1 of J such that $U(x_1, B_1)$ is contained in U. Index the elements of B_1 so that

$$B_1 = \{\alpha_1, \ldots, \alpha_{n_1}\}.$$

Now suppose that x_k and n_k are given, and that α_j is defined for $j = 1, \ldots, n_k$. Let B_k denote the set

$$B_k = \{\alpha_j \mid 1 \leq j \leq n_k\}.$$

Define a point x_{k+1} of X by setting

$$x_{k+1}(\alpha_j) = j \quad \text{for } 1 \leq j \leq n_k, \text{ and}$$
$$x_{k+1}(\alpha) = 1 \quad \text{for all other } \alpha.$$

Then x belongs to P_1. Choose B_{k+1} so that $U(x_{k+1}, B_{k+1})$ is contained in U. Without loss of generality, we can choose B_{k+1} so that it properly contains B_k. Index the elements of $B_{k+1} - B_k$ so that

$$B_{k+1} - B_k = \{\alpha_j \mid n_k < j \leq n_{k+1}\}.$$

By induction (actually, recursive definition), we have defined x_i and α_i and n_i for all i.

Step 2. Now, define a point y of X by setting

$$y(\alpha_j) = j \quad \text{for all } j,$$
$$y(\alpha) = 2 \quad \text{for all other } \alpha.$$

Then y belongs to P_2. Choose C so that $U(y, C)$ is contained in V. Since C is finite, it contains α_j for only finitely many j; choose n_k so that C contains no α_j for which $j > n_k$. We shall show that $U(y,C)$ intersects $U(x_{k+1}, B_{k+1})$, so that the sets U and V are not disjoint.

Let us define a point z of X (cleverly!) by setting

$$z(\alpha_j) = j \quad \text{for } 1 \leq j \leq n_k,$$
$$z(\alpha_j) = 1 \quad \text{for } n_k < j \leq n_{k+1}, \text{ and}$$
$$z(\alpha) = 2 \quad \text{for all other } \alpha.$$

Then $z(\alpha_j)$ equals $x_{k+1}(\alpha_j)$ for $1 \leq j \leq n_{k+1}$, so that z belongs to $U(x_{k+1}, B_{k+1})$. On the other hand, we show that $z(\alpha) = y(\alpha)$ for α in C, so that z belongs to $U(y,C)$; our result is then proved. It is certainly true that $z(\alpha) = y(\alpha)$ if α is one of the indices α_j, for in that case $j \leq n_k$, so that $z(\alpha_j) = j = y(\alpha_j)$. And it is true that $z(\alpha) = y(\alpha)$ if α is not one of the indices α_j; for in that case $z(\alpha) = 2 = y(\alpha)$. □

<u>Theorem F.2.</u> There is a space that is regular but not completely regular.

<u>Proof.</u> The proof follows the outline given in Exercise 11 of §33.

<u>Step 1.</u> Given an even integer m, Let L_m denote the line segment $m \times [-1,0]$ in the plane. And given an odd integer n, and an integer $k \geq 2$, let $C_{n,k}$ denote the union of the line segments

$$(n + (k-1)/k) \times [-1,0]$$
$$(n - (k-1)/k) \times [-1,0]$$

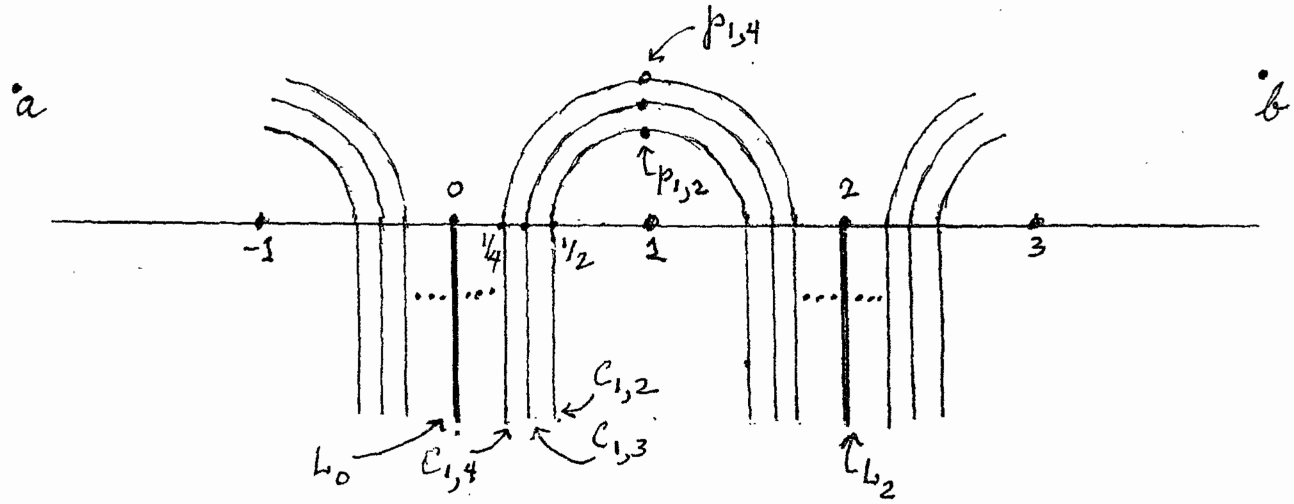
and the semicircle

$$\{x \times y \mid (x-n)^2 + y^2 = (k-1)^2/k^2 \quad \text{and} \quad y \geq 0\}$$

in the plane. We call $C_{n,k}$ an "arch" and we call L_m a "pillar." Finally, we let X be the union of the pillars L_m, for all even integers m, and the arches $C_{n,k}$, for all odd integers n and all integers $k \geq 2$, along with two additional points a and b, which we call the "points at infinity." For each odd n and each $k \geq 2$, we let $p_{n,k}$ be the point

$$p_{n,k} = n \times (k-1)/k ;$$

it is the "peak" of the arch $C_{n,k}$. See the accompanying figure.



We now topologize X in a most unusual fashion. We take as basis elements all sets of the following five types:

(i) Each one-point set $\{p\}$, where p is a point lying on any one of the arches $C_{n,k}$ that is different from the peak $p_{n,k}$ of this arch.

(ii) The set formed from one of the sets $C_{n,k}$ by deleting finitely many points.

(iii) For each even integer m, each ϵ with $0 < \epsilon < 1$, and each $y \in [-1,0]$, the intersection of X with the horizontal open line segment $(m - \epsilon, m + \epsilon) \times y$.

(iv) For each even integer m, the union of $\{a\}$ and the set of points $x \times y$ of X for which $x < m$.

(v) For each even integer m, the union of $\{b\}$ and the set of points $x \times y$ of X for which $x > m$.

The basis elements of type (ii) are the neighborhoods of the peaks; those of type (iii) are the neighborhoods of points lying on the pillars; and those of types (iv) and (v) are the neighborhoods of the points at infinity. It is easy (but boring) to check the conditions for a basis; we leave it to you. Each of the arches $C_{n,k}$ is an open set of X.

We shall call the space X "Thomas' arches," because it was invented by the topologist John Thomas.

Step 2. It is trivial to check that X is a T_1-space; given two points, each has a neighborhood that excludes the other. To check regularity, let p be a point of X, and let U be a basis element containing p. We consider several cases, showing there is a neighborhood V of p such that $\bar{V} \subset U$.

If U is a basis element of types (i), (ii), or (iii), then $\bar{U} = U$, and we are finished. So suppose that U is of type (iv), consisting of the point a along with those points $x \times y$ of X for which $x < m$. If p is the point a, then we let V consist of the point a along with those points $x \times y$ of X for which $x < m - 2$. Then $\bar{V} = V \cup L_{m-2}$, which lies in U. If p is some other point of U, there is a basis element V of type (i), (ii), or (iii) containing p and lying in U; then $\bar{V} = V$ and we are finished. The argument when U is of type (v) is similar.

<u>Step 3.</u> X is not completely regular. Indeed, we show that if f is any continuous function $f: X \to [0,1]$, then $f(a) = f(b)$.

Given n,k, let $S_{n,k}$ be the set of points p of the arch $C_{n,k}$ for which the value of f at p is different from the value of f at the peak $p_{n,k}$ of the arch. Then the set $S_{n,k}$ is countable: Let $f(p_{n,k}) = c$. The set $f^{-1}(c)$ is a G_δ-set in X, since it is the intersection of the open sets $f^{-1}((c - \frac{1}{n}, c + \frac{1}{n}))$. Each of these open sets contains all but finitely many points of $C_{n,k}$. Hence their intersection contains all but countably many points of $C_{n,k}$. Thus $S_{n,k}$ is countable.

It follows that the union of all the sets $S_{n,k}$ is countable. Therefore we may choose a real number d with $-1 \leq d \leq 0$ such that the horizontal line $\mathbb{R} \times \{d\}$ intersects <u>none</u> of the sets $S_{n,k}$. This means that for each arch $C_{n,k}$, the value of f at the points where the arch intersects this horizontal line equals the value of f at the peak of the arch.

Now for each even integer m, let c_m be the point where the line $\mathbb{R} \times \{d\}$ intersects the pillar L_m. We assert that the values of f at the points c_m and c_{m+2} are equal.

To prove this fact, set $n = m+1$, consider the arch $C_{n,k}$, and let a_k and b_k denote the points of intersection of this arch with the line $\mathbb{R} \times \{d\}$. (For convenience, let a_k be the one with smaller x-coordinate.) Then as k increases, the sequence a_k converges to c_m, while the sequence b_k converges to c_{m+2}. Continuity of f then implies that $f(a_k)$ converges to $f(c_m)$ and $f(b_k)$ converges to $f(c_{m+2})$. But by construction,

$$f(a_k) = f(p_{n,k}) = f(b_k)!$$

We conclude that $f(c_m) = f(c_{m+2})$.

It follows that the values of f at the points c_m are all equal. But c_m converges to the point a as m goes to $-\infty$, and c_m converges to b as m goes to $+\infty$. It follows from continuity of f that $f(a) = f(b)$. $\square$