The Prüfer Manifold.

The so-called Prüfer manifold is a space that is locally 2-euclidean and Hausdorff, but not normal. In discussing it, we follow the outline of Exercise 6 on p. 317.

Definition. Let A be the following subspace of $\mathbb{R}^2$:

$$A = \{(x,y) \mid x > 0\}.$$

Given a real number c, let B_c be the following subspace of $\mathbb{R}^3$:

$$B_c = \{(x,y,c) \mid x \leq 0\}.$$

Let X be the set that is the union of A and all the spaces B_c, for c real. Topologize X by taking as a basis all sets of the following three types:

(i) U, where U is open in A.

(ii) V, where V is open in the subspace of B_c consisting of points with $x < 0$.

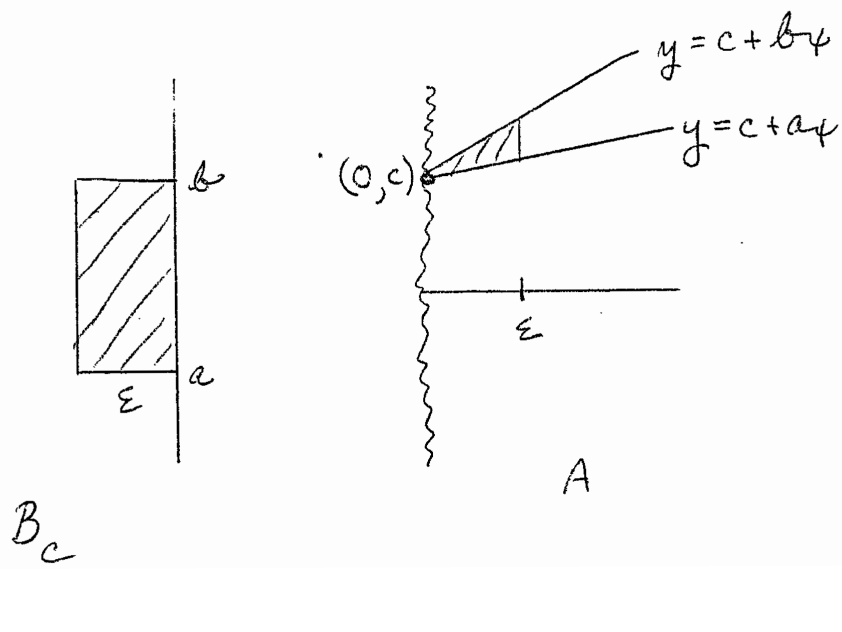
(iii) For each open interval $I = (a,b)$ of $\mathbb{R}$, each real number c, and each $\varepsilon > 0$, the set $A_c(I,\varepsilon) \cup B_c(I,\varepsilon) = U_c(I,\varepsilon)$, where

$$A_c(I,\varepsilon) = \{(x,y) \mid 0 < x < \varepsilon \text{ and } c + ax < y < c + bx\},$$

$$B_c(I,\varepsilon) = \{(x,y,c) \mid -\varepsilon < x \leq 0 \text{ and } a < y < b\}.$$

The space X is called the "Prüfer manifold."

Let us sketch what the basis elements of type (iii) look like. Given c, I, and ε, the basis $U_c(I,\varepsilon)$ is the union of the two shaded figures in the figure.



It is easy to check that these sets form a basis for a topology. The intersection of a set of type (i) with any other basis element is empty or is open in A, and the intersection of a set of type (ii) with any other basis element is empty or is open in the subspace $x<0$ of B_c. Finally, the intersection of the sets $U_c(I,\varepsilon)$ and $U_d(I',\varepsilon')$ is open in A if $c \neq d$; it is empty if $c = d$ and I is disjoint from I'; and finally if $c = d$ and I intersects I', it equals the set

$$U_c(I \cap I', \min(\varepsilon, \varepsilon')).$$

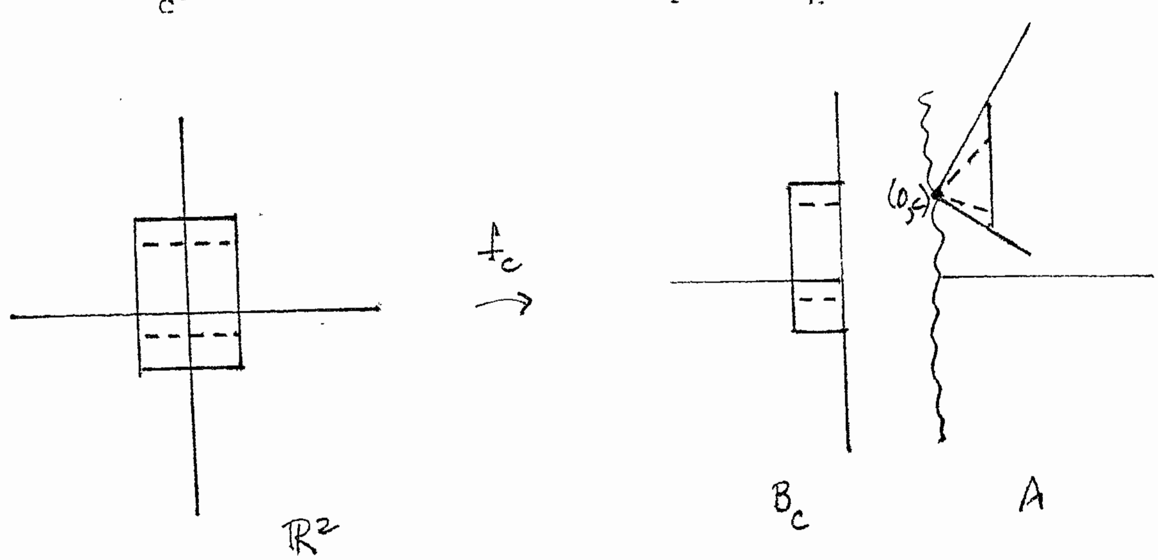
We show that X is locally 2-euclidean. For fixed c, the set $B_c \cup A$ is a union of basis elements, and therefore is open in X. Surprisingly, it is actually homeomorphic to $\mathbb{R}^2$! Consider the map $f_c : \mathbb{R}^2 \to X$ given by

$$f_c(x,y) = (x,y,c) \quad \text{for} \quad x \leq 0,$$
$$f_c(x,y) = (x, c + xy) \quad \text{for} \quad x > 0.$$

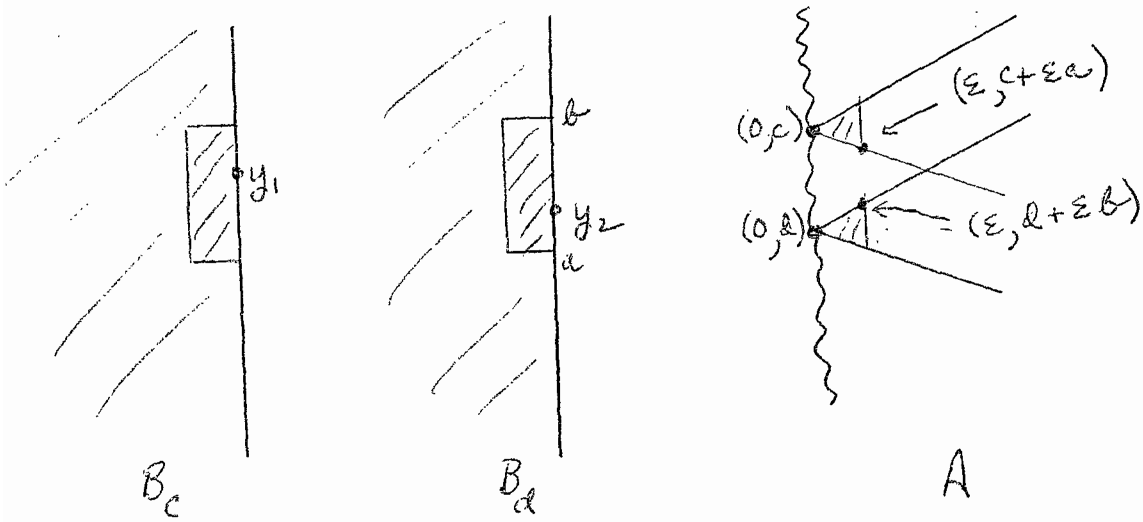
The map f_c carries the subspace of $\mathbb{R}^2$ consisting of points (x,y) with $x \leq 0$ bijectively onto B_c, and it carries the subspace consisting of points with $x > 0$ bijectively onto A (Each vertical line $x = x_0$ is carried bijectively onto itself).

To show that f_c is a homeomorphism, we note that we can take as basis for $\mathbb{R}^2$ all open sets lying the the half-plane $x<0$, all open sets lying in the half-plane $x>0$, and all open sets of the form $(-\varepsilon, \varepsilon) \times (a,b)$. Each of these is mapped by f_c onto one of the basis elements for X; and conversely. [The open set $(-\varepsilon, \varepsilon) \times (a,b)$ is mapped onto $U_c((a,b), \varepsilon)$.] The map f_c is pictured in the accompanying figure.

It follows that X is locally 2-euclidean, since it is covered by the open sets $B_c \cup A$, each of which is homeomorphic to $\mathbb{R}^2$.



We show that X is Hausdorff. The only case where some care is required is the case where the two distinct points are points of the "edges" of the half-spaces B_c. If they belong to the same half-space B_c, then they are of the form $(0,y_1,c)$ and $(0,y_2,c)$. In this case, we need merely choose disjoint intervals I_1 and I_2 about y_1 and y_2, respectively; then the basis elements $U_c(I_1, \varepsilon)$ and $U_c(I_2, \varepsilon)$ are disjoint (for any ε). Now consider two points of the form $(0,y_1,c)$ and $(0,y_2,d)$, where $c \neq d$. Choose an open interval $I = (a,b)$ containing both y_1 and y_2. Then if ε is sufficiently small, the basis elements $U_c(I, \varepsilon)$ and $U_d(I, \varepsilon)$ are disjoint. [See the accompanying figure. Assuming $d < c$, one chooses ε so that $\varepsilon(b-a) < (c-d)$, so that $d + \varepsilon b < c + \varepsilon a$.]



Finally, we show that X is not normal. The proof follows a familiar pattern. Let L be the subspace of X consisting of all points of the form $(0,0,c)$. It is closed in X; and it has the discrete topology since each basis element of type (iii) intersects L in at most a single point. One show repeats the argument given in Example 3 of §31, which showed that $\mathbb{R}_\ell^2$ is not normal. If X were normal, then for every subset C of L, one could choose disjoint open sets U_C and V_C of X containing C and $L - C$, respectively. Letting D be the set of points of A having rational coordinates, one defines $\theta : \mathcal{P}(L) \rightarrow \mathcal{P}(D)$ by setting

$$\Theta(C) = D \cap U_C ,$$
$$\Theta(\emptyset) = \emptyset ,$$
$$\Theta(L) = D.$$

One shows readily that Θ is injective; then one derives a contradiction from cardinality considerations, since L is uncountable and D is countable. □