18.906: Problem Set III

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet.

Extra credit for finding mistakes and telling me about them early!

11. **(a)** Embed $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$ as the vectors with last coordinate equal to 0. This defines embeddings $S^{n-1} \hookrightarrow S^n$ and also $\mathbf{R}P^{n-1} \hookrightarrow \mathbf{R}P^n$. Write S^∞ and $\mathbf{R}P^\infty$ for the spaces obtained as the direct limit of these sequences.

(i) Define a CW structure on each sphere making S^{n-1} the $(n-1)$-skeleton of S^∞, and on each projective space making $\mathbf{R}P^{n-1}$ the $(n-1)$-skeleton of $\mathbf{R}P^\infty$.

(ii) Show that the double covers $S^{n-1} \to \mathbf{R}P^{n-1}$ combine to give a double cover $S^\infty \to \mathbf{R}P^\infty$.

(iii) Determine the homotopy groups of these two spaces.

(b) Carry out the analogous story when $\mathbf{R}$ is replaced by $\mathbf{C}$.

12. Let $\omega \in \pi_1(S^1 \vee S^2)$ and $\alpha \in \pi_2(S^1 \vee S^2)$ be represented by the inclusion of the two spheres into the wedge. Form a new CW complex X by attaching a 3-cell by means of a map representing the homotopy class $2\alpha - \omega \cdot \alpha \in \pi_2(S^1 \vee S^2)$. Show that the inclusion of S^1 into X induces isomorphisms in π_1 and in homology, but that X is not weakly equivalent to the circle.

[So no simple adjustment to the Whitehead theorem will work. Notice however that the map on universal covers is not an isomorphism in homology.]

13. Identify each of the spaces $\tau_{>2}\mathbb{C}P^n$ and $\tau_{\leq 2}S^2$ with known CW complexes (up to homotopy type, of course).

14. **(a)** Let $N < G$ be a normal subgroup, with quotient group H. Show that there is a fibration $K(G,1) \to K(H,1)$ with fiber weakly equivalent to – a phrase we'll neglect below! – $K(N,1)$.

(b) Suppose that G is abelian. Then the same argument gives us a fibration $K(G,n) \to K(H,n)$ with fiber $K(N,n)$. But show also that there is a fibration $K(N,n) \to K(G,n)$ with fiber $K(H,n-1)$, and a fibration $K(H,n) \to K(N,n+1)$ with fiber $K(G,n)$.

For example, what is the homotopy fiber of the map $\mathbb{C}P^\infty \to \mathbb{C}P^\infty$ represented by twice a generator of $H^2(\mathbb{C}P^\infty)$?

15. Let Y be a simple space and N an integer, and suppose that $N\pi_*(Y) = 0$. Let (X, A) be a relative CW complex and assume that $H_*(X, A; \mathbb{F}_p) = 0$ whenever the prime p divides N. Show that the restriction map $[X, Y] \to [A, Y]$ is bijective.

18.906 Algebraic Topology II
Spring 2020