# Chapter 5

# Characteristic classes, Steenrod operations, and cobordism

## 33 Chern classes, Stiefel-Whitney classes, and the Leray-Hirsch theorem

A good supply of interesting geometric objects is provided by the theory of principal $G$-bundles, for a topological group $G$. For example giving a principal $GL_n(\mathbb{C})$-bundle over $X$ is the same thing as giving a complex $n$-plane bundle over $X$.

Principle bundles reflect a great deal of geometric information in their topology. This is a great asset, but it can make them correspondingly hard to visualize. It's reasonable to hope to construct invariants of principal $G$-bundles of some more understandable sort. A good candidate is a cohomology class.

So let's fix an integer $n$ and an abelian group $A$, and try to associate, in some way, a class $c(\xi) \in H^n(Y; A)$ to any principal $G$-bundle $\xi$ over $Y$. To make this useful, this association should be *natural*: given $f : X \to Y$ and a principal $G$-bundle $\xi$ over $Y$, we can pull $\xi$ back under $f$ to a principal $G$-bundle $f^*\xi$ over $X$, and find ourselves with two classes in $H^n(X; A)$: $f^*c(\xi)$ and $c(f^*(\xi))$. Naturality insists that these two classes coincide. This means, incidentally, that $c(\xi)$ depends only on the isomorphism class of $\xi$. Let $\mathrm{Bun}_G(X)$ denote the set of isomorphism classes of principal $G$-bundles over $X$; it is a contravariant functor of $X$. We have come to the definition:

**Definition 33.1.** Let $G$ be a topological group, $A$ an abelian group, and $n \geq 0$. A *characteristic class* for principal $G$-bundles with values in $H^n(-; A)$ is a natural transformation of functors **Top** $\to$ **Set**:

$$c : \mathrm{Bun}_G(X) \to H^n(X; A)\,.$$

Cohomology classes are more formal or algebraic, and are correspondingly relatively easy to work with. $\mathrm{Bun}_G(X)$ is often hard (or impossible) to compute, partly because it has no algebraic structure and partly exactly because its elements are interesting geometrically, while $H^n(X; A)$ is relatively easy to compute but its elements are not very geometric. A characteristic class provides a bridge between these two, and information flows across this bridge in both directions. It gives computable information about certain interesting geometric objects, and provides a geometric interpretation of certain formal or algebraic things.

**Example 33.2.** The Euler class is the first and most fundamental characteristic class. Let $R$ be a commutative ring. The Euler class takes an $R$-oriented real $n$-plane bundle $\xi$ and produces an

$n$-dimensional cohomology class $e(\xi)$, given by the transgression of the class in $H^0(B; H^{n-1}(\mathbb{S}\xi))$ that evaluates to 1 on every orientation class. Naturality of the Gysin sequence shows that this assignment is natural. There are really only two cases: $R = \mathbb{Z}$ and $R = \mathbb{F}_2$. A $\mathbb{Z}$-orientation of a vector bundle is the same thing as an orientation in the usual sense, and the Euler class is a natural transformation

$$e : \mathrm{Vect}_n^{or}(X) = \mathrm{Bun}_{SO(n)}(X) \to H^n(X;\mathbb{Z})\,.$$

Any vector bundle is canonically $\mathbb{F}_2$-oriented, so the mod 2 Euler class is a natural transformation

$$e : \mathrm{Vect}_n(X) = \mathrm{Bun}_{O(n)}(X) \to H^n(X;\mathbb{F}_2)\,.$$

On CW complexes, $\mathrm{Bun}_G(-)$ is representable: there is a "universal" principal $G$-bundle $\xi_G : EG \downarrow BG$ such that

$$[X, BG] \to \mathrm{Bun}_G(X)\,, \quad f \mapsto f^*\xi_G$$

is a bijection. A characteristic classes $\mathrm{Bun}_G(-) \to H^n(-;A)$ is the same thing as a class in $H^n(BG;A)$, or, since cohomology is also representable, as a homotopy class of maps $BG \to K(A,n)$.

Thus for example set of all integral characteristic classes of complex line bundles is given by $H^*(BU(1)) = \mathbb{Z}[e]$. Is there an analogous classification of characteristic classes for higher dimensional complex bundles? How about real bundles?

### Chern classes

We'll begin with complex vector bundles. Any complex vector bundle (numerable of course) admits a Hermitian metric, well defined up to homotopy. This implies that $\mathrm{Bun}_{U(n)}(X) \to \mathrm{Bun}_{GL_n(\mathbb{C})}(X)$ is bijective; $BU(n) \to BGL_n(\mathbb{C})$ is a homotopy equivalence. I will tend to favor using $U(n)$ and $BU(n)$.

A finite dimensional complex vector space $V$ determines an orientation of the underlying real vector space: Pick an ordered basis $(e_1, \ldots, e_n)$ for $V$ over $\mathbb{C}$, and provide $V$ with the ordered basis over $\mathbb{R}$ given by $(e_1, ie_1, \ldots, e_n, ie_n)$. The group $\mathrm{Aut}_{\mathbb{C}}(V)$ acts transitively on the space of complex bases. But choosing a basis for $V$ identifies $\mathrm{Aut}(V)$ with $GL_n(\mathbb{C})$, which is path connected. So the set of oriented real bases obtained in this way are all in the same path component of the set of all oriented real bases, and hence defines an orientation of $V$.

This construction yields a natural transformation $\mathrm{Vect}_{\mathbb{C}}(-) \to \mathrm{Vect}_{\mathbb{R}}^{or}(-)$. In particular, the real 2-plane bundle underlying a complex line bundle has a preferred orientation – the determined in each fiber $\xi_x$ by $(v, iv)$ where $v \neq 0$ in $\xi_x$. A complex line bundle $\xi$ over $B$ thus has a well-defined Euler class $e(\xi) \in H^2(B;\mathbf{Z})$.

**Theorem 33.3** (Chern classes)**.** *There is a unique family of characteristic classes for complex vector bundles that assigns to a complex $n$-plane bundle $\xi$ over $X$ its* $k$th Chern class $c_k^{(n)}(\xi) \in H^{2k}(X;\mathbf{Z})$, *$k \in \mathbb{N}$, such that:*

- $c_0^{(n)}(\xi) = 1$.
- $c_1^{(1)}(\xi) = -e(\xi)$.
- *The* Whitney sum formula *holds: if $\xi$ is a $p$-plane bundle and $\eta$ is a $q$-plane bundle, then*

$$c_k^{(p+q)}(\xi \oplus \eta) = \sum_{i+j=k} c_i^{(p)}(\xi) \cup c_j^{(q)}(\eta) \;\in\; H^{2k}(X;\mathbf{Z}).$$

*Moreover, if* $\xi_n$ *is the universal* $n$*-plane bundle, then*

$$H^*(BU(n);\mathbf{Z})\cong\mathbf{Z}[c_1^{(n)},\ldots,c_n^{(n)}]$$

*where* $c_k^{(n)}=c_k^{(n)}(\xi_n)$.

This result says that all characteristic classes for complex vector bundles are given by polynomials in the Chern classes, and that there are no universal algebraic relations among the Chern classes. (Shiing-Shen Chern (1911–2004) was a father of twentieth century differential geometry, and a huge force in the development of mathematics in China.)

**Remark 33.4.** Since $BU(n)$ supports the universal $n$-plane bundle $\xi_n$, the Chern classes $c_k^{(n)}=c_k^{(n)}(\xi_n)$ are themselves universal, pulling back to the Chern classes of any other $n$-plane bundle.

The $(p+q)$-plane bundle $\xi_p\times\xi_q=\mathrm{pr}_1^*\xi_p\oplus\mathrm{pr}_2^*\xi_q$ over $BU(p)\times BU(q)$ is classified by a map $\mu: BU(p)\times BU(q)\to BU(p+q)$. The Whitney sum formula computes the effect of $\mu$ on cohomology:

$$\mu^*(c_k^{(n)})=\sum_{i+j=k}c_i^{(p)}\times c_j^{(q)}\in H^{2k}(BU(p)\times BU(q))\,,$$

where, you'll recall, $x\times y=\mathrm{pr}_1^*x\cup\mathrm{pr}_2^*y$.

The Chern classes are "stable" in the following sense. Let $\epsilon$ be the trivial one-dimensional complex vector bundle over $X$ and let $\xi$ be an $n$-dimensional vector bundle over $X$. What is $c_k^{(n+q)}(\xi\oplus q\epsilon)$? The trivial bundle is obtained by pulling back under $X\to *$:

$$\begin{array}{ccc} X\times\mathbf{C}^q=E(q\epsilon) & \longrightarrow & \mathbf{C}^q \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

By naturality, we find that $c_j^{(n)}(n\epsilon)=0$ for $j>0$. The Whitney sum formula therefore implies that

$$c_k^{(n+q)}(\xi\oplus q\epsilon)=c_k^{(n)}(\xi).$$

Thus the Chern class only depends on the "stable equivalence class" of the vector bundle. Also, the map $BU(n-1)\to BU(n)$ classifying $\xi_{n-1}\oplus\epsilon$ sends $c_k^{(n)}$ to $c_k^{(n-1)}$ for $k<n$ and $c_n^{(n)}$ to 0.

For this reason, we will drop the superscript on $c_k^{(n)}(\xi)$, and simply write $c_k(\xi)$.

## Grothendieck's construction

Let $\xi: E\xrightarrow{p}X$ be a complex $n$-plane bundle. Associated to it is a fiber bundle whose fiber over $x\in X$ is $\mathbb{P}(p^{-1}(x))$, the projective space of the vector space given by the fiber of $\xi$ over $x$. This "projectivization" can also be described using the $GL_n(\mathbb{C})$ action on $\mathbf{C}P^{n-1}=\mathbb{P}(\mathbb{C}^n)$ induced from its action on $\mathbb{C}^n$, and forming the balanced product

$$\mathbb{P}(\xi)=P\times_{GL_n(\mathbb{C})}\mathbf{C}P^{n-1}$$

where $P\downarrow X$ is the principalization of $\xi$.

Let us attempt to compute the cohomology of $\mathbb{P}(\xi)$ using the Serre spectral sequence:

$$E_2^{s,t}=H^s(X;H^t(\mathbf{C}P^{n-1}))\Rightarrow H^{s+t}(\mathbb{P}(\xi)).$$

We claim that this spectral sequence almost completely determines the cohomology of $\mathbb{P}(\xi)$ as a ring. Here is a general theorem that tells us what to look for, and what we get.

**Theorem 33.5** (Leray-Hirsch). *Let $\pi : E \to B$ be a fibration and $R$ a commutative ring. Assume that $B$ is path connected, so that the fiber is well defined up to homotopy. Call it $F$, and suppose that for each $t$ the $R$-module $H^t(F)$ is free of finite rank. Finally, assume that the restriction $H^*(E) \to H^*(F)$ is surjective. (One says that the fibration is "totally non-homologous to zero.") Because $H^t(F)$ is a free $R$-module for each $t$, the surjection $H^*(E) \to H^*(F)$ admits a splitting; pick one, say $s : H^*(F) \to H^*(E)$. The projection map renders $H^*(E)$ a module over $H^*(B)$. The $H^*(B)$-linear extension of $s$,*

$$\overline{s} : H^*(B) \otimes_R H^*(F) \to H^*(E)$$

*is then an isomorphism of $H^*(B)$-modules.*

*Proof.* First we claim that the group $\pi_1(B)$ acts trivially on the cohomology of $F = \pi^{-1}(*)$. The map of fibrations

$$\begin{array}{ccc} E & \xrightarrow{1} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ B & \longrightarrow & * \end{array}$$

shows that the map $H_*(F) \to H_*(E)$ is equivariant with respect to the group homomorphisms $\pi_1(B) \to \pi_1(*)$. In cohomology, this says that the restriction $H^*(E) \to H^*(F)$ has image in the $\pi_1(B)$-invariant subgroup (which, by the way, is $H^0(B; H^*(F))$). So the assumption that this map is surjective guarantees that the action of $\pi_1(B)$ on $H_*(F)$ is trivial.

Now the edge homomorphism in the Serre spectral sequence

$$E_2^{s,t} = H^s(B; H^t(F)) \underset{s}{\Longrightarrow} H^{s+t}(E)$$

is that restriction map. Our assumption that $H^t(F)$ is free of finite rank implies that

$$E_2^{s,t} = H^s(B) \otimes_R H^t(F)$$

as $R$-algebras. All the generators lie on either $t = 0$ or $s = 0$. The ones on the base survive because the differentials hit zero groups. The generators on the fiber survive by assumption. So inductively you find that $E_r = E_{r+1}$, and hence that the entire spectral sequence collapses at $E_2$.

We now define a new filtration on $H^*(E)$ with the advantage that it is a filtration by $H^*(B)$-modules. I call it the "Quillen filtration," though it is probably older. It's the *increasing* filtration given by

$$F_t H^n(E) = F^{n-t} H^n(E)\,.$$

For instance, $F_0 H^n(E) = F^n H^n(E) = \mathrm{im}(H^n(B) \to H^n(E)) \cong H^n(B)$; or

$$F_0 H^*(E) = \mathrm{im}(H^*(B) \to H^*(E))\,.$$

On the level of associated graded modules,

$$\mathrm{gr}_t H^n(E) = F^{n-t} H^n(E) / F^{n-t+1} H^n(E) = E_\infty^{n-t,t}$$

– that is, the $t$th row: so

$$\mathrm{gr}_t H^*(E) = E_\infty^{*,t} = E_2^{*,t} = H^*(B) \otimes H^t(F)$$

Now we can think about the map $\overline{s} : H^*(B) \otimes H^*(F) \to H^*(E)$. Filter $H^*(B) \otimes H^*(F)$ by degree in $H^*(F)$:

$$F_t(H^*(B) \otimes H^*(F)) = H^*(B) \otimes \bigoplus_{i \le t} H^i(F)\,.$$

The map $\overline{s}$ respects filtrations and is an isomorphism on associated graded modules: so it is an isomorphism. □

Returning now to the example of the projectivization of a vector bundle, $\mathbb{P}(\xi) \downarrow X$, the hypotheses of the Leray-Hirsch Theorem are satisfied except perhaps surjectivity of the restriction to the fiber.

Here's where the representation of a cohomology class as a characteristic class comes in useful. The cohomology of the fiber over $x \in X$ is generated as an $R$-module by powers of the Euler class of the canonical line bundle $\lambda_x$ over $\mathbb{P}(\xi_x)$. Since $i^* : H^*(E) \to H^*(\mathbb{C}P^{n-1})$ is an $R$-algebra map, it will suffice to see that $e(\lambda_x)$ is in the image of $i^*$. Since the Euler class is natural, the natural thing to do is to construct a line bundle over the whole of $\mathbb{P}(\xi)$ that restricts to $\lambda_x$ on $\xi_x$. And indeed these line bundles over fibers assemble themselves into a tautologous line bundle, call it $\lambda$, over $\mathbb{P}(\xi)$.

So we have an expression for $H^*(\mathbb{P}(\xi))$ as a module over $H^*(X)$:

$$H^*(\mathbb{P}(\xi)) = H^*(X)\langle 1, e, e^2, \ldots, e^{n-1}\rangle .$$

where $e = e(\lambda) \in H^2(\mathbb{P}(\xi))$. This gives us some information about the algebra structure in $H^*(\mathbb{P}(\xi))$, but not complete information. What is lacking is an expression for $e^n$ in terms of the basis given by lower powers of $e$. The Euler class $e$ satisfies a unique monic polynomial equation $c_\xi(e) = 0$, where $c_\xi(t)$ is the "Chern polynomial"

$$c_\xi(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-1} t + c_n .$$

with $c_k \in H^{2k}(X)$.

The naturality of this construction guarantees that the $c_k$'s are natural in the $n$-plane bundle $\xi$; they are characteristic classes. We will see that they satisfy the axioms for Chern classes set out above.

Note that the Whitney sum formula has a nice expression in terms of the Chern polynomials:

$$c_\xi(t) c_\eta(t) = c_{\xi\oplus\eta}(t) .$$

## Stiefel-Whitney classes

Exactly parallel theorems hold for real $n$-plane bundles, with mod 2 coefficients:

**Theorem 33.6** (Stiefel-Whitney classes)**.** *There is a unique family of characteristic classes for real vector bundles that assigns to a real $n$-plane bundle $\xi$ over $X$ its "$k$th Stiefel-Whitney class" $w_k(\xi) \in H^{2k}(X; \mathbb{F}_2)$, $k \in \mathbb{N}$, such that:*

- $w_0(\xi) = 1$.
- *If $n = 1$ then $w_1(\xi) = e(\xi)$.*
- *The* Whitney sum formula *holds: if $\xi$ is a $p$-plane bundle and $\eta$ is a $q$-plane bundle, then*

$$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \cup w_j(\eta) \in H^{2k}(X; \mathbb{F}_2).$$

*Moreover, if $\xi_n$ is the universal $n$-plane bundle, then*

$$H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, \ldots, w_n]$$

*where $w_k = w_k(\xi_n)$.*

And the same construction produces them:

$$H^*(\mathbb{P}(\xi);\mathbb{F}_2) = H^*(B;\mathbb{F}_2)[e]/(e^n + w_1 e^{n-1} + \cdots + w_{n-1}e + w_n)$$

for unique elements $w_i \in H^i(B;\mathbb{F}_2)$.

**Remark 33.7.** The Euler class depends only on the sphere bundle of the vector bundle $\xi$, but these constructions depend heavily on the existence of an underlying vector bundle. This is a genuine dependence in the case of Chern classes, but it turns out that the Stiefel-Whitney classes depend only on the sphere bundle. We'll explain this a little while.

**Remark 33.8.** In the complex case, the triviality of the local coefficient system can be verified in other ways as well. After all, the action of $\pi_1(X)$ on the fiber $H^*(\mathbb{C}P^{n-1})$ is compatible with the action of $\pi_1(BU(n))$ on the homology of the fiber of the projectivized universal example. But since $U(n)$ is connected, its classifying space is simply connected.

You can't make this argument in the real case, but then you don't have to since we are looking at an action of $\pi_1(B)$ on a one-dimensional vector space over $\mathbb{F}_2$.

**Example 33.9.** Complex projective space $\mathbf{C}P^n$ is a complex manifold, and its tangent bundle is thereby endowed with a complex structure. A standard argument shows that

$$\tau_{\mathbf{C}P^n} = \mathrm{Hom}(\lambda, \lambda^\perp)\,.$$

Adding $\epsilon = \mathrm{Hom}(\lambda, \lambda)$, we find

$$\tau_{\mathbf{C}P^n} \oplus \epsilon = (n+1)\lambda\,.$$

Thus by the Whitney sum formula

$$c_\tau(t) = c_{\tau\oplus\epsilon}(t) = c_\lambda(t)^{n+1} = (1-e)^{n+1}$$

and so

$$c_k(\tau_{\mathbf{C}P^n}) = (-1)^k \binom{n+1}{k} e^k\,.$$

## 34 $H^*(BU(n))$ and the splitting principle

Here's another description of the Chern classes.

**Theorem 34.1.** *Let $n \geq 1$. There is a unique family of characteristic classes $c_i(\xi) \in H^{2i}(B(\xi))$, $1 \leq i \leq n$, for complex $n$-plane bundles $\xi$ such that if $\xi$ is isomorphic to $\zeta \oplus (n-i)\epsilon$ then*

$$c_i(\xi) = (-1)^i e(\zeta)$$

*where $e(\zeta)$ is the Euler class of the oriented real $2i$-bundle underlying $\zeta$. These classes generate all characteristic classes for $n$-plane bundles and there are no universal algebraic relations among them.*

We will prove this by computing the cohomology of $BU(n)$, by induction on $n$. Here's how $BU(n)$ and $BU(n-1)$ are related. Embed $U(n-1) \hookrightarrow U(n)$ by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

This subgroup is exactly the set of matrices fixing the last basis vector $e_n$ in $\mathbb{C}^n$. The orbit of $e_n$ is the subspace $S^{2n-1}$ of unit vectors in $\mathbb{C}^n$, which is thus identified with the homogeneous space $U(n)/U(n-1)$.

Make a choice of $EU(n)$ – a contractible on which $U(n)$ acts principally – the Stiefel model $V_n(\mathbb{C}^\infty)$ for example. The orbit space is then the Grassmann model for $BU(n)$. The subgroup $U(n-1)$ also acts principally on $EU(n)$, so we get a model for $BU(n-1)$:

$$BU(n-1) = EU(n)/U(n-1) = (EU(n) \times_{U(n)} U(n))/U(n-1) = \\ EU(n) \times_{U(n)} (U(n)/U(n-1)) = EU(n) \times_{U(n)} S^{2n-1}\,.$$

This establishes $p : BU(n-1) \to BU(n)$ as the unit sphere bundle in the universal complex $n$-plane bundle $\xi_n$. The map $BU(n-1) \to BU(n)$ classifies the $n$-plane bundle $\xi_{n-1} \oplus \epsilon$.

Here's a restatement of Theorem 34.1 in terms of universal examples.

**Theorem 34.2.** *There exist unique classes $c_i \in H^{2i}(BU(n))$ for $1 \le i \le n$ such that:*

1. *the map $p_* : H^*(BU(n)) \to H^*(BU(n-1))$ sends*

$$c_i \mapsto \begin{cases} c_i & \textit{for} \quad i < n \\ 0 & \textit{for} \quad i = n\,. \end{cases}$$

2. *the Euler class $e$ of the oriented real $2n$-plane bundle underlying the universal complex $n$-plane bundle $\xi_n$ is related to the top class $c_n$ by the equation*

$$c_n = (-1)^n e \in H^{2n}(BU(n))\,.$$

*Moreover,*

$$H^*(BU(n)) = \mathbf{Z}[c_1, \ldots, c_n]\,.$$

We postpone the verification that the classes we constructed in the last lecture coincide with these.

*Proof.* We will study the Gysin sequence of the spherical fibration

$$S^{2n-1} \to BU(n-1) \xrightarrow{p} BU(n)\,.$$

For a general oriented spherical fibration

$$S^{2n-1} \to E \xrightarrow{p} B$$

the Gysin sequence takes the form

$$\cdots \to H^{q-1}(E) \xrightarrow{p_*} H^{q-2n}(B) \xrightarrow{e\cdot} H^q(B) \xrightarrow{p^*} H^q(E) \xrightarrow{p_*} H^{q-2n+1}(B) \to \cdots\,.$$

where $e \in H^{2n}(B)$ is the Euler class.

Suppose we know that $H^*(E)$ vanishes in odd dimensions. Then either the source or the target of each instance of the Umkher map $p_*$ is zero, so we receive a short exact sequence

$$0 \to H^{q-2n}(B) \xrightarrow{e\cdot} H^q(B) \xrightarrow{p^*} H^q(E) \to 0\,.$$

This shows several things:

- $e \in H^{2n}(B)$ is a non-zero-divisor;
- $p^*$ is surjective and induces an isomorphism $H^*(B)/(e) \to H^*(E)$;
- $p^*$ is an isomorphism in dimensions less than $2n$;
- $H^q(B) = 0$ for $q$ odd.

The last is clear for $q < 2n$, but feeding this into the leftmost term we find by induction that $H^q(B) = 0$ for all odd $q$.

Now let's suppose in addition that $H^*(E)$ is a polynomial algebra. Lift the generators to elements in $H^*(B)$. (If they all happen to lie in dimension less than $2n$, these lifts are unique.) Extending to a map of algebras gives a map $H^*(E) \to H^*(B)$. Further adjoining $e$ gives us an algebra map

$$H^*(E)[e] \to H^*(B)$$

which when composed with $p^*$ kills $e$ and maps $H^*(E)$ by the identity. We claim this map is an isomorphism. To see this, filter both sides by powers of $e$. Modulo $e$ this map is an isomorphism from what we observed above. On both sides, multiplication by $e$ induces an isomorphism from one associated quotient to the next, so the map induces an isomorphism on associated graded modules. The five-lemma shows that it induces an isomorphism mod $e^k$ for any $k$. But the powers of $e$ increase in dimension, so we obtain an isomorphism in each dimension.

These observations provide the inductive step. All that remains is to start the induction. We can, if we like, use what we know about $H^*(\mathbf{C}P^\infty)$ and start with $n = 2$, though starting at $n = 1$ makes sense too, and provides another perspective on the computation of $H^*(\mathbf{C}P^\infty)$.

We define $c_n \in H^{2n}(BU(n))$ to be $(-1)^n e(\xi_n)$, also a generator. The choice of sign will make it agree with our earlier definition. □

Once we verify that these classes coincide with the classes constructed in the last lecture, we will have available an important interpretation of the top Chern class: up to sign it is the Euler class of the underlying oriented real vector bundle.

## The splitting principle

A wonderful fact about Chern classes is that it suffices to check relations among them on sums of line bundles. This is captured by the following theorem.

**Theorem 34.3** (Splitting principle). *Let $\xi : E \downarrow X$ be a complex $n$-plane bundle. There exists a map $f : \mathrm{Fl}(\xi) \to X$ such that:*

1. *$f^*\xi \cong \lambda_1 \oplus \cdots \oplus \lambda_n$, where the $\lambda_i$ are line bundles on $\mathrm{Fl}(\xi)$, and*
2. *the map $f^* : H^*(X) \to H^*(\mathrm{Fl}(\xi))$ is monic.*

*Proof.* We have already done the hard work, in our study of the projectivization $\pi : \mathbb{P}(\xi) \to X$. We found that the Serre spectral sequence collapses at $E^2$. This implies that the projection map induces a monomorphism in cohomology. We used the "tautologous" line bundle $\lambda$ on $\mathbb{P}(\xi)$. The key additional point about this construction is that there is a canonical embedding $\lambda \hookrightarrow \pi^*\xi$ of vector bundles over $\mathbb{P}(\xi)$. A vector in $E(\lambda)$ is $(v \in L \subseteq \xi_x)$ (where $L$ is a line in the fiber $\xi_x$). A vector in the pullback $\pi^*\xi$ is $(v \in \xi_x, L \subseteq \xi_x)$.

By picking a metric on $\xi$ we see that when pulled back to $\mathbb{P}(\xi)$ a line bundle splits off. Now just induct (using our important standing assumption that vector bundles have finite dimensional fibers). □

It's worth being more explicit about what this "flag bundle" $\mathrm{Fl}(\xi)$ is. The complement of $\lambda$ in $\pi^*\xi$ over $\mathbb{P}(\xi)$ is the the space of vectors of the form $(v \in L^\perp, L \subseteq \xi_x)$. If we iterated this construction, we will get, in the end, the space of ordered orthogonal decompositions of fibers into lines. This can be built as a balanced product. Let $\mathrm{Fl}_n$ be the space of "orthogonal flags," that is, decompositions of $\mathbb{C}^n$ into an ordered sequence of $n$ 1-dimensional subspaces. There is an evident action of $U(n)$ on this space, and

$$\mathrm{Fl}(\xi) = P \times_{U(n)} \mathrm{Fl}_n$$

where $P \downarrow X$ is the principal $U(n)$ bundle associated to $\xi$ (and a choice of Hermitian metric).

The action of $U(n)$ on $\mathrm{Fl}_n$ is transitive, and the isotropy subgroup of $(\mathbb{C}e_1, \ldots, \mathbb{C}e_n)$ is the subgroup of diagonal unitary matrices,

$$T^n = (S^1)^n \subseteq U(n)\,,$$

so

$$\mathrm{Fl}_n = U(n)/T^n\,.$$

In the universal case, over $BU(n)$,

$$\mathrm{Fl}(\xi_n) = EU(n) \times_{U(n)} (U(n)/T^n) = EU(n)/T^n = BT^n$$

and this is just a product of $n$ copies of $\mathbb{C}P^\infty$. So we have discovered that

$$H^*(BU(n)) \hookrightarrow H^*(BT^n) = \mathbb{Z}[t_1, \cdots, t_n]$$

where $t_i$ is the Euler class of the line bundle $\mathrm{pr}_i^*\lambda$, the pull back of the universal line bundle under the projection onto the $i$th factor of $\mathbb{C}P^\infty$. What is the image?

Well, the symmetric group $\Sigma_n$ sits inside the unitary group as matrices with a single 1 in each column. The maximal torus $T^n$ is sent to itself by conjugation by a permutation matrix, which has the effect of reordering the diagonal entries. In cohomology, the action permutes the generators. These permutation matrices also act by conjugation on all of $U(n)$, but there they act trivially on $H^*(BU(n))$ since any matrix is connected to the identity matrix by a path in $U(n)$. The consequence is that the image of $H^*(BU(n))$ lies in the symmetric invariants:

$$H^*(BU(n)) \hookrightarrow H^*(BT^n)^{\Sigma_n}\,.$$

These symmetric invariants are well-studied in Algebra! Define the *elementary symmetric polynomials* $\sigma_i$ as the coefficients in the product of $t - t_i$'s:

$$\prod_{i=1}^{n}(t - t_i) = \sum_{j=0}^{n} \sigma_j t^{n-j}$$

For example,

$$\sigma_0 = 1 \quad , \quad \sigma_1 = -\sum_{j=1}^{n} t_j \quad , \quad \sigma_n = (-1)^n \prod_{j=1}^{n} t_j\,.$$

The theorem from algebra is that the elementary symmetric polynonomials are algebraically independent and generate the ring of symmetric invariants –

$$R[t_1, \ldots, t_n]^{\Sigma_n} = R[\sigma_1, \ldots, \sigma_n]$$

– over any coefficient ring $R$.

If we give each $t_i$ a grading of 2, the elementary symmetric polynomials are homogeneous and $|\sigma_i| = 2i$.

So $H^*(BU(n))$ embeds into a graded algebra of exactly the same size. This does not yet show that the embedding is surjective! For each $q$, we know that $H^q(BU(n))$ embeds into $H^q(BT^n)$ as a subgroup of the same rank. If $L$ is a free abelian group of finite rank and $L'$ is a subgroup, the little exact sequence

$$0 \to \mathrm{Tor}_1(L/L', \mathbb{F}_p) \to L' \otimes \mathbb{F}_p \to L \otimes \mathbb{F}_p$$

shows that the $p$-torsion in $L/L'$ vanishes if $L' \otimes \mathbb{F}_p \to L \otimes \mathbb{F}_p$ is injective. Now our argument above actually works for any coefficient ring, so $H^*(BU(n); \mathbb{F}_p) \to H^*(BT^n; \mathbb{F}_p)$ is monic for any prime $p$. Because $H^*(BU(n))$ is torsion free this says that $H^*(BU(n)) \otimes \mathbb{F}_p \to H^*(BT^n) \otimes \mathbb{F}_p$ is monic for any prime. The result is that the index of $H^*(BU(n))$ in $H^*(BT^n)^{\Sigma_n}$ is prime to $p$ for every prime number $p$, and so this injection must also be surjective.

We have proven most of:

**Theorem 34.4.** *The inclusion $T^n \hookrightarrow U(n)$ induces an isomorphism*

$$H^*(BU(n)) \xrightarrow{\cong} H^*(BT^n)^{\Sigma_n} .$$

*Under this identification, the classes $c_i$ constructed in Theorem 2 map to the elementary symmetric functions.*

In the context of Chern classes, the elements $t_i$ are called "Chern roots." The extension $H^*(BU(n)) \hookrightarrow H^*(BT^n)$ adjoins the roots of the Chern polynomial

$$c(t) = t^n + c_1 t^{n-1} + \cdots + c_n$$

**Remark 34.5.** Everything we have done admits a version for real vector bundles, with mod 2 coefficients. One point deserves some special attention: the argument we gave for why conjugation by a permutation induces the identity on $H^*(BU(n))$ fails because the group $O(n)$ is not path-connected. However, there is a better and more general argument available.

**Lemma 34.6.** *Let $G$ be any topological group and $g \in G$. The self-map of $BG$ induced by conjugation by $g$ is homotopic to the identity.*

*Proof.* The proof is an easy exercise using the material from Lecture 21. We regard $G$ as a topological category with one object. Conjugation induces an endofunctor $c_g$. A natural transformation from the identity to $c_g$ is given by the morphism $g$:

$$\begin{array}{ccc} * & \xrightarrow{g} & * \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle c_g(h)=ghg^{-1}} \\ * & \xrightarrow{g} & * . \end{array}$$

And natural transformations induce homotopies. □

Of course the map $c_g : BG \to BG$ is not homotopic to the identity through *basepoint preserving* homotopies! On $\pi_1(BG) = \pi_0(G)$ it induces conjugation by $[g] \in \pi_0(G)$.

# 35 The Thom class and Whitney sum formula

We now have four perspectives on Chern classes:

1. Axiomatic
2. Grothendieck's definition in terms of $H^*(\mathbb{P}(\xi))$
3. In terms of Euler classes
4. As elementary symmetric polynomials via the splitting principle

In this lecture we will explain why these are four facets of the same gem, though at the expense of introducing a new perspective on the Euler class. Developing that perspective lets us introduce another important construction in topology, the Thom space. We'll use that to verify that (3) and (4) agree. Then we'll prove the Whitney sum formula from this perspective. We'll take the identification of Chern classes with symmetric polynomials as the starting point.

## Thom space and Thom class

Let $\xi : E \xrightarrow{p} B$ be a real $n$-plane bundle. The *Thom space* is obtained by forming the one-point compactification of each fiber, and then identifying all the newly adjoined basepoints to a single point. If $B$ is a compact Hausdorff space, this amounts to the one-point compactification of the total space $E(\xi)$.

**Example 35.1.** There is a canonical homeomorphism

$$\mathrm{Th}(\lambda^* \downarrow \mathbb{R}P^{n-1}) \to \mathbb{R}P^n .$$

It is given by sending $(\varphi \in L^*, L \subseteq \mathbb{R}^n)$ to the graph of $\varphi$ in $\mathbb{R}^n \times \mathbb{R}$. This map embeds $E(\lambda^*)$ into $\mathbb{R}P^n$, and misses only the line $\mathbb{R}e_{n+1}$. This establishes $\mathbb{R}P^n$ as the one-point compactification of $E(\lambda^*)$. (It also shows that $\lambda^*$ is the normal bundle of the linear embedding $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$.)

By choosing a metric we get a different expression for the same space. Let $\mathbb{D}(\xi)$ and $\mathbb{S}(\xi) = \partial\mathbb{D}(\xi)$ denote the unit disk and unit sphere bundles. The Thom space of $\xi$ is the quotient space

$$\mathrm{Th}(\xi) = \mathbb{D}(\xi)/\mathbb{S}(\xi) .$$

Rather than this quotient space, you may prefer to think of the pair $(\mathbb{D}(\xi), \mathbb{S}(\xi))$; it is homotopy equivalent to the pair $(E(\xi), E(\xi)\backslash Z)$, where $Z$ is the image of the zero-section.

Note that $\mathrm{Th}(0) = B/\varnothing = B_+$, the base with a disjoint basepoint adjoined. The Thom space of the $n$-plane bundle over a point is $D^n/\partial D^n = S^n$.

An important point about the Thom space construction is its behavior on the product of two bundles, say $\xi$ and $\eta$. Since

$$\partial(D^p \times D^q) = (\partial D^p \times D^q) \cup (D^p \times \partial D^q) ,$$

we find

$$\mathrm{Th}(\xi \times \eta) = \frac{\mathbb{D}(\xi \times \eta)}{\partial\mathbb{D}(\xi \times \eta)} = \frac{\mathbb{D}(\xi) \times \mathbb{D}(\eta)}{\mathbb{S}(\xi) \times \mathbb{D}(\eta) \cup \mathbb{D}(\xi) \times \mathbb{S}(\eta)} = \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) .$$

In particular, if $\eta$ is the $n$-plane bundle over a point, $\xi \times \eta = \xi \oplus n\epsilon$ and

$$\mathrm{Th}(\xi \oplus n\epsilon) = \mathrm{Th}(\xi) \wedge S^n = \Sigma^n \,\mathrm{Th}(\xi) .$$

In general, the Thom space is a "twisted $n$-fold suspension."

The Thom space construction is natural for bundle maps: Given $f : B' \to B$, covered by a bundle map $\xi' \to \xi$ (so that $\xi' \cong f^*\xi$) we get a canonical pointed map

$$\overline{f} : \mathrm{Th}(\xi') \to \mathrm{Th}(\xi)\,.$$

This construction can be used to define a relative product in the cohomology of the Thom space, in the following way. Notice that the bundle $0 \times \xi$ over $B \times B$ is just the pullback of $\xi$ under $\mathrm{pr}_2 : B \times B \to B$. The diagonal map $\Delta : B \to B \times B$ satisfies $\mathrm{pr}_2 \circ \Delta = 1_B$, and is therefore covered by a bundle map $\xi \to 0 \times \xi$, which then induces a twisted diagonal map

$$\mathrm{Th}(\xi) \to \mathrm{Th}(0) \wedge \mathrm{Th}(\xi) = B_+ \wedge \mathrm{Th}(\xi)\,.$$

This in turn induces a "relative cup product" in cohomology:

$$\cup : H^*(B) \otimes \overline{H}^*(\mathrm{Th}(\xi)) \to \overline{H}^*(\mathrm{Th}(0) \wedge \mathrm{Th}(\xi)) \to \overline{H}^*(\mathrm{Th}(\xi))\,.$$

Since the diagonal map is associative and unital, this map defines on $\overline{H}^*(\mathrm{Th}(\xi))$ the structure of a module over the graded ring $H^*(B)$.

Here is the essential fact about the Thom space.

**Proposition 35.2** (Thom isomorphism theorem). *Let $R$ be a commutative ring and let $\xi$ be an $R$-oriented real $n$-plane bundle over $B$. There is a unique class $U \in \overline{H}^n(\mathrm{Th}(\xi); R)$ that restricts on each fiber to the dual of the orientation class, and the map*

$$- \cup U : H^*(B) \to \overline{H}^*(\mathrm{Th}(\xi))$$

*is an isomorphism.*

*Proof.* The proof is very simple, if you grant yet another relative form of the Serre spectral sequence. This time I want to have a fibration $p : E \to B$ – say a fiber bundle – together with a subbundle $p_0 : E_0 \to B$. Then there is spectral sequence

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-), p_0^{-1}(-))) \underset{s}{\Longrightarrow} H^{s+t}(E, E_0)$$

We will apply this to the fiber bundle pair $(\mathbb{D}(\xi), \mathbb{S}(\xi))$. The fiber pair is then $(D^n, S^{n-1})$, which has cohomology in just one dimension! This spectral sequence has just one row: the $n$th row. It collapses at $E_2$, there are no extension problems, and we get a canonical isomorphism

$$H^s(B; H^n(p^{-1}(-), p_0^{-1}(-))) \to H^{s+n}(\mathbb{D}(\xi), \mathbb{S}(\xi)) = \overline{H}^{s+n}(\mathrm{Th}(\xi))\,.$$

The assumed orientation identifies the local coefficient system with the constant system $R$. The generator of $E_2^{0,n}$ survives to a class $U$ that restricts as stated, and the multiplicative structure of the spectral sequence implies that this is an isomorphism of modules over $H^*(B)$. □

## Thom and Euler

We now use this construction to define a new class in $H^n(B)$ associated to the oriented $n$-plane bundle $\xi$, by means of the composite

$$\pi : B \rightleftarrows \mathbb{D}(\xi) \to \mathrm{Th}(\xi)\,.$$

The first map is the zero-section, homotopy inverse to the projection map. The second one is the collapse map. The Thom class $U \in \overline{H}^n(\mathrm{Th}(\xi))$ pulls back under this map to a class in $\overline{H}^n(B)$.

This class is at least up to sign the Euler class as we defined it earlier:

**Lemma 35.3.** *This class coincides up to sign with the Euler class: $\pi^* U = \pm e$.*

It is easy to verify at least that they generate the same subgroup (which proves that they are the same with coefficients in $\mathbb{F}_2$). Work in the universal case. As a notational choice, we will work over $\mathbb{Z}$, so we are looking at $\xi_n$ over $BSO(n)$. We've seen that the total space of its sphere bundle is $BSO(n-1)$. The Serre spectral sequence for this fibration shows that the kernel of the projection map $p^* : H^n(BSO(n-1)) \to H^n(BSO(n))$ is the image of the transgression $H^{n-1}(S^{n-1}) \to H^n(BSO(n-1))$. So the kernel is cyclic and generated by the Euler class. On the other hand, we have the cofibration sequence

$$BSO(n-1) \to BSO(n) \xrightarrow{\pi} MSO(n)$$

where we are using Thom's notation $MSO(n) = \mathrm{Th}(\xi_n)$. The Thom class $U \in H^n(MSO(n))$ generates this group (by the Thom isomorphism theorem) so its image in $H^n(BSO(n))$ also generates $\ker(H^n(BSO(n)) \to H^n(BSO(n-1)))$.

We will see, as a consequence of a computation of $H^*(BSO(n); \mathbf{Z}[1/2])$, that this kernel is infinite cyclic if $n$ is even, so then the generator is at least well defined up to sign. For homework you will show that $2e = 0$ if $n$ is even, so the generator is then unique.

But in fact, it's better just to take $\pi^* U$ as the *definition* of the Euler class. With that definition, we get a new construction of the Gysin sequence: It is the long exact cohomology sequence of the pair $(\mathrm{Th}(\xi), B)$, aided by the Thom isomorphism:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{s-1}(B) & \xrightarrow{p^*} & H^{s-1}(E) & \xrightarrow{\delta} & \overline{H}^s(\mathrm{Th}(\xi)) & \xrightarrow{\pi^*} & H^s(B) & \xrightarrow{p^*} & H^s(E) & \longrightarrow \cdots \\
 & & & & \searrow^{p_*} & \cong \uparrow {-\cup U} & \nearrow^{\cdot e} & & & & \\
 & & & & & H^{s-n}(B) . & & & & &
\end{array}$$

This is a long exact sequence of modules over $H^*(B)$. This gives a different perspective on integration along the fiber:

$$(p_* x) \cup U = \delta x .$$

We'll just use this definition going forward. Notice that with this definition, the Euler class is multiplicative for Whitney sum. We should be careful about orientations. The direct sum of oriented vector spaces $V$ and $W$ has an orientation given by putting a positive ordered basis for $V$ first and follow it by a positive ordered basis for $W$. This convention orients the Whitney sum of two vector bundles over a space.

**Proposition 35.4.** *Let $\xi$ and $\eta$ be oriented vector bundles over spaces $X$ and $Y$.*

$$e(\xi \times \eta) = e(\xi) \times e(\eta) .$$

*Proof.* First, $U_\xi \wedge U_\eta \in \overline{H}^{p+q}(\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta))$ is a Thom class for $\xi \times \eta$, since it restricts on fiber pairs to the direct sum orientation. Then the collapse maps are compatible:

$$\begin{array}{ccc}
\mathrm{Th}(\xi \times \eta) & \xrightarrow{=} & \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) \\
\uparrow {\scriptstyle \pi_{X \times Y}} & & \uparrow {\scriptstyle \pi_X \wedge \pi_Y} \\
(X \times Y)_+ & \xrightarrow{=} & X_+ \wedge Y_+
\end{array}$$

commutes, and in cohomology we chase

$$\begin{array}{ccc} U_\xi \wedge U_\eta & \longmapsto & U_{\xi\times\eta} \\ \downarrow & & \downarrow \\ e(\xi)\times e(\eta) & \longmapsto & e(\xi\times\eta)\,. \end{array}$$

□

If we take $X = Y$ here and pull back along the diagonal, $\xi \times \eta$ goes to the Whitney sum and $e(\xi) \times e(\eta)$ goes to the cup-product:

$$e(\xi \oplus \eta) = e(\xi) \cdot e(\eta)\,.$$

### Euler class and symmetric polynomials

One of our descriptions of the Chern classes was this: If an $n$-plane bundle $\xi$ splits $\zeta \oplus (n-k)\epsilon$, then $c_k(\xi) = (-1)^k e(\zeta)$. Let's check that this holds for the classes defined by means of elementary symmetric functions. It might be clearest if we look at the universal example, where the splitting map $f : BT^n \to BU(n)$ pulls $\xi_n$ back to the direct sum of line bundles $\lambda_1 \oplus \cdots \oplus \lambda_n$ and induces an isomorphism $f^* : H^*(BU(n)) \to H^*(BT^n)^{\Sigma_n}$. Let's do the case $k = n$ first, so I want to show that $(-1)^n e(\xi_n)$ maps to $\sigma_n$. Using multiplicativity of the Euler class,

$$f^* e(\xi_n) = e(\lambda_1 \oplus \cdots \oplus \lambda_n) = e(\lambda_1)\cdots e(\lambda_n)\,.$$

With the notation $t_i = e(\lambda_i)$, this shows that

$$f^*((-1)^n e(\xi_n)) = (-1)^n t_1 \cdots t_n = \sigma_n\,.$$

For smaller $k$, we'll use the fact that the maximal tori $T^k \subseteq U(k)$ are compatible as $k$ increases. This gives the commutative diagram

$$\begin{array}{ccc} H^{2k}(BU(n)) & \longrightarrow & H^{2k}(BT^n)^{\Sigma_n} \\ \downarrow & & \downarrow \\ H^{2k}(BU(k)) & \longrightarrow & H^{2k}(BT^k)^{\Sigma_k} \end{array}$$

The elementary symmetric polynomial definition of $c_k$ specifies that it maps to $\sigma_k$ along the top arrow. We want to see that this class maps to $(-1)^k e(\xi_k) \in H^{2k}(BU(k))$. Well, by the $k = n$ case that we just did, we know that that class maps to $\sigma_k$ along the bottom. So what remains is to check that $\sigma_k \in H^{2k}(BT^n)^{\Sigma_n}$ maps to the class of the same name in $H^{2k}(BT^k)^{\Sigma_k}$.

To keep things straight, let's write $\sigma_k^{(n)}$ for the first class and $\sigma_k^{(k)}$ for the second. The restriction $H^*(BT^n) \to H^*(BT^k)$ sends $t_i$ to $t_i$ if $i \le k$ and to 0 if $i > k$. So

$$\begin{array}{ccc} \sum_{i=0}^n \sigma_i^{(n)} t^{n-i} & = & \prod_{j=1}^n (t - t_j) \\ & & \downarrow \\ \left(\sum_{i=0}^k \sigma_i^{(k)} t^{k-i}\right) t^{n-k} & = & \left(\prod_{j=1}^k (t-t_j)\right) t^{n-k} \end{array}$$

and comparing coefficients we see that $\sigma_k^{(n)} \mapsto \sigma_k^{(k)}$.

### The Whitney sum formula

By our discussion above, the Whitney sum formula of Theorem 33.3 reduces to proving the following identity:

$$\sigma_k^{(p+q)} = \sum_{i+j=k} \sigma_i^{(p)} \cdot \sigma_j^{(q)} \tag{5.1}$$

inside $\mathbf{Z}[t_1, \ldots, t_p, t_{p+1}, \ldots, t_{p+q}]$. Here, $\sigma_i^{(p)}$ is thought of as a polynomial in $t_1, \ldots, t_p$, while $\sigma_j^{(q)}$ is thought of as a polynomial in $t_{p+1}, \ldots, t_{p+q}$. To derive Equation (5.1), simply compare coefficients in the following:

$$\begin{aligned}
\sum_{k=0}^{p+q} \sigma_k^{(p+q)} t^{p+q-k} &= \prod_{i=1}^{p+q} (t - t_i) \\
&= \prod_{i=1}^{p} (x - t_i) \cdot \prod_{j=p+1}^{p+q} (t - t_j) \\
&= \left( \sum_{i=0}^{p} \sigma_i^{(p)} t^{p-i} \right) \left( \sum_{j=0}^{q} \sigma_j^{(p)} t^{q-j} \right) \\
&= \sum_{k=0}^{p+q} \left( \sum_{i+j=k} \sigma_i^{(p)} \sigma_j^{(q)} \right) t^{p+q-k} .
\end{aligned}$$

Hassler Whitney once called this his hardest theorem. Apparently he didn't have the splitting principle working for him.

## 36 Closing the Chern circle, and Pontryagin classes

### Back to Grothendieck

Now we'll use the splitting principle to show that the Chern classes (defined as corresponding to the elementary symmetric polynomials) participate in a monic polynomial satisfied by the Euler class of the tautologous bundle over the projectivization of a vector bundle. This will complete the identification of the various versions of Chern classes.

So we have an $n$-plane bundle $\xi$ over $B$, and consider the projectivization $\pi : \mathbb{P}(\xi) \to B$. We observed in the last lecture that the tautologous bundle $\lambda$ embeds (canonically) into the pullback $\pi^*\xi$. Let $\overline{\lambda}$ denote the complex conjugate or inverse line bundle, so that $\overline{\lambda} \otimes \lambda = \epsilon$. Tensoring $\pi^*\xi$ with $\overline{\lambda}$ thus results in a bundle with a trivial summand; that is, with a nowhere vanishing section. Its Euler class therefore vanishes. We will compute what that Euler class is, using the splitting principle.

The splitting principle allows us to assume that $\xi$ is a sum of line bundles, say $\xi = \lambda_1 \oplus \cdots \oplus \lambda_n$. Then

$$\overline{\lambda} \otimes \pi^*\xi = \bigoplus_{i=1}^{n} \overline{\lambda} \otimes \pi^*\lambda_i \,.$$

By multiplicativity of the Euler class, we find

$$e(\overline{\lambda} \otimes \pi^*\xi) = \prod_{i=1}^{n} e(\overline{\lambda} \otimes \pi^*\lambda_i) \,.$$

Write $t$ for $e(\lambda) \in H^2(\mathbb{P}(\xi))$, so that $e(\overline{\lambda}) = -t$. Also write $t_i = e(\lambda_i)$, so that

$$e(\overline{\lambda} \otimes \pi^*\lambda_i) = \pi^* t_i - t$$

and

$$e(\overline{\lambda} \otimes \pi^*\xi) = \prod_{i=1}^{n} (\pi^* t_i - t) = (-1)^n \sum_{j=0}^{n} (\pi^* c_j(\xi)) t^{n-j} \, .$$

Since $e(\overline{\lambda} \otimes \pi^*\xi) = 0$, this shows that our new Chern classes satisfy the identity Grothendieck used to define them. Since these coefficients were unique, this identifies Grothendieck's definition with the others we have introduced.

### Stiefel-Whitney classes

Same story! Well, almost. We don't have the even/odd argument working for us anymore. We want to know that the Euler class is a non-zero-divisor. We do have the splitting principle, which assures us that

$$f^* : H * (BO(n); \mathbb{F}_2) \hookrightarrow H^*(BC_2^n; \mathbb{F}_2)^{\Sigma_n} \, .$$

By multiplicativity of the Euler class, it maps to $t_1 \cdots t_n \in H^n(BC_2^n; \mathbb{F}_2)$, which is nonzero in this integral domain and so is a non-zero-divisor. The result:

**Proposition 36.1.** $H^*(BO(n); \mathbb{F}_2) = \mathbb{F}_2[w_1, \ldots, w_n]$.

While we are talking about Stiefel-Whitney classes, let me point out that $w_1 \in H^1(B; \mathbb{F}_2)$ is precisely the obstruction to orientability of $\xi : E \downarrow B$. If $B$ is path-connected, it can be identified with the homomorphism $\pi_1(B) \to C_2$ that takes on the value $-1$ on $\sigma$ if the orientation of the fiber is reversed under the homotopy endomorphism of the fiber given by $\sigma$. You can check this in the universal case: The class $w_1 \in H^1(BO(n); \mathbb{F}_2)$ is represented by a map $BO(n) \to K(\mathbb{F}_2, 1)$. This map is the bottom Postnikov stage of $BO(n)$, and its homotopy fiber is the simply connected Whitehead cover of $BO(n)$. We know what that is, since $SO(n) \hookrightarrow O(n)$ is the connected component of the identity (and is the kernel of $\det : O(n) \to C_2$).

The map $BSO(n) \to BO(n)$ is (at least homotopy theoretically) a double cover; the fiber is $S^0$, so we are entitled to a Gysin sequence. The Euler class of this spherical fibration is exactly $w_1$, a non-zero-divisor, so we discover the short exact sequence

$$0 \to H^*(BO(n); \mathbb{F}_2) \xrightarrow{e\cdot} H^*(BO(n); \mathbb{F}_2) \to H^*(BSO(n); \mathbb{F}_2) \to 0 \, .$$

This shows that $H^*(BSO(n); \mathbb{F}_2)$ is the polynomial algebra on the images of $w_2, \ldots, w_n$:

$$H^*(BSO(n); \mathbb{F}_2) = \mathbb{F}_2[w_2, \ldots, w_n] \, .$$

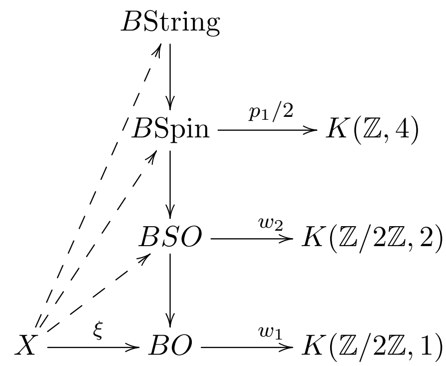
It often happens that one cares about only the "stable" equivalence class of a vector bundle. This leads one to consider the direct limit or union

$$BO = \lim_{n \to \infty} BO(n) \, .$$

Its cohomology is given by

$$H^*(BO) = \mathbb{F}_2[w_1, w_2, \ldots], .$$

Of course the limit of the $BSO(n)$'s is written $BSO$. It is the simply-connected cover of $BO$. It's interesting to contemplate the rest of the Whitehead tower of $BO$. For a while the spaces involved have names:




## Pontryagin classes

Real vector bundles have integral characteristic classes too! They were studied by Lev Pontryagin (1908–1988, Steklov Institute, blinded in an accident at age 14). The idea is to use Chern classes to define such things. Given a real vector bundle $\xi$ we can tensor up to the complex vector bundle $\mathbb{C} \otimes_{\mathbb{R}} \xi$, and study its Chern classes.

Complex vector bundles arising in this way have some additional structure. Any complex vector bundle $\zeta : E \downarrow B$ has a "complex conjugate" vector bundle $\overline{\zeta}$ with the same underlying real vector bundle but with complex structure defined by letting $z \in \mathbb{C}$ act on $\overline{\zeta}$ the way $\overline{z}$ acted on $\zeta$. We've already seen this construction for line bundles, when $\lambda \otimes \overline{\lambda} = \epsilon$.

The complexification $\mathbb{C} \otimes_{\mathbb{R}} \xi$ of a real vector bundle comes equipped with an isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \xi \cong \overline{\mathbb{C} \otimes_{\mathbb{R}} \xi}$$

given by $z \otimes v \mapsto \overline{z} \otimes v$. We discover that

$$c_i(\mathbb{C} \otimes_{\mathbb{R}} \xi) = c_i(\overline{\mathbb{C} \otimes_{\mathbb{R}} \xi}) \, ,$$

so we should ask: What are the Chern classes of the complex conjugate of a complex vector bundle?

**Lemma 36.2.** $c_i(\overline{\xi}) = (-1)^i c_i(\xi)$ .

*Proof.* Exercise; use any one of the perspectives on Chern classes that we have developed. □

This puts no restriction on $c_i(\mathbb{C} \otimes_{\mathbb{R}} \xi)$ for $i$ even, but forces $2c_i(\mathbb{C} \otimes_{\mathbb{R}} \xi) = 0$ for $i$ odd. The 2-torsion will get in the way, so let's work with coefficients in a ring $R$ in which 2 is invertible – a $\mathbb{Z}[1/2]$-algebra, such as $\mathbb{Z}[1/2]$ itself, or $\mathbb{F}_p$ for $p \neq 2$. We already have Stiefel-Whitney classes with mod 2 coefficients, so this is not so bad.

**Definition 36.3.** The $k$th *Pontryagin class* of a real vector bundle $\xi$ is

$$p_k(\xi) = (-1)^k c_{2k}(\mathbb{C} \otimes_{\mathbb{R}} \xi) \in H^{4k}(X; R) \, .$$

Of course $p_k(\xi) = 0$ if $k > n/2$, since $\xi \otimes \mathcal{C}$ is of complex dimension $n$. The strange sign does not interfere with the Whitney sum formula:

$$p_k(\xi \oplus \eta) = (-1)^k \sum_{i+j=k} c_{2i}(\mathbb{C} \otimes_{\mathbb{R}} \xi) c_{2j}(\mathbb{C} \otimes_{\mathbb{R}} \eta) = \sum_{i+j=k} p_i(\xi) p_j(\eta)$$

since the odd terms contribute only 2-torsion, which we have eliminated by working over a $\mathbb{Z}[1/2]$-algebra.

The Pontryagin classes are defined for vector bundles, orientable or not. They are independent of the orientation if there is one. But an oriented $2k$-plane bundle over $B$ has an Euler class $e(\xi) \in H^{2k}(B)$, and we might ask how it is related to the Pontryagin classes. The sign is there in the definition of the Pontryagin classes so that the following important relation is satisfied.

**Lemma 36.4.** *For any oriented $2k$-plane bundle, $p_k(\xi) = e(\xi)^2$.*

*Proof.* We need to be careful about orientations. We have the isomorphism of *real* vector bundles

$$\xi \oplus \xi \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{R}} \xi \,,$$

defined $(v, w) \mapsto v + iw$. We have established an orientation on $\mathbb{C} \otimes_{\mathbb{R}} \xi$. But suppose that $\xi$ itself came equipped with an orientation. This puts an orientation on the direct sum. How are the two orientations related to each other? If $e_1, \dots, e_n$ is a positive basis for an ordered vector space $V$, then we are comparing the ordered bases

$$\begin{aligned} e_1, e_2, \dots, e_n, ie_1, ie_2, \dots, ie_n &\quad \text{for} \quad V \oplus V \quad \text{and} \\ e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n &\quad \text{for} \quad \mathbb{C} \otimes_{\mathbb{R}} V \,. \end{aligned}$$

Relating them requires

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$$

transpositions, so they give the same orientation if this number is even and opposite orientations if it is odd.

Now we can compute:

$$\begin{aligned} p_k(\xi) &= (-1)^k c_{2k}(\mathbb{C} \otimes_{\mathbb{R}} \xi) = (-1)^k e(\mathbb{C} \otimes_{\mathbb{R}} \xi) \\ &= (-1)^k (-1)^{2k(2k-1)/2} e(\xi \oplus \xi) = e(\xi)^2 \end{aligned}$$

since $2k(2k-1)/2 \equiv k \mod 2$. □

We can now systematically compute the cohomology of $BSO(n)$ away from 2 by induction on $n$ using the Gysin sequence. Here's the result.

**Theorem 36.5.** *With coefficients in any $\mathbb{Z}[1/2]$-algebra, the cohomology of $BSO(n)$ is polynomial for all $n$. When $n = 2k + 1$, the generators are $p_1, \dots, p_k$. When $n = 2k$, the generators are $p_1, \dots, p_{k-1}, e_n$. The maps $H^*(BSO(n)) \to H^*(BSO(n-1)$ take Pontryagin classes to themselves, except that $H^{4k}(BSO(2k+1)) \to H^{4k}(BSO(2k))$ sends $p_k$ to $e_{2k}^2$.*

Here's a table of the algebra generators, with the squares of the Euler classes added in to indicate how $p_k$ restricts.

| | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $H^*(BSO(2))$ | $e_2$ | $(e_2^2)$ | | | | |
| $H^*(BSO(3))$ | | $p_1$ | | | | |
| $H^*(BSO(4))$ | | $p_1, e_4$ | | $(e_4^2)$ | | |
| $H^*(BSO(5))$ | | $p_1$ | | $p_2$ | | |
| $H^*(BSO(6))$ | | $p_1$ | $e_6$ | $p_2$ | | $(e_6^2)$ |
| $H^*(BSO(7))$ | | $p_1$ | | $p_2$ | | $p_3$ |

We can then compute $H^*(BO(n);R)$ for $R$ a $\mathbb{Z}[1/2]$-algebra by using the fiber sequence

$$BSO(n) \to BO(n) \to \mathbb{R}P^\infty .$$

The spectral sequence has $E_2^{s,t} = H^s(\mathbb{R}P^\infty; H^t(BSO(n)))$. There are local coefficients here, but with any local coefficients the higher cohomology of $\mathbb{R}P^\infty$ is killed by 2 and so vanishes for us. As a result the edge homomorphism

$$H^*(BO(n);R) \to H^*(BSO(n);R)^{C_2}$$

is an isomorphism. The generator of $\pi_1(\mathbb{R}P^\infty)$ tracks the effect of reversing orientations: it fixes the Pontryagin classes and negates the Euler classes. The result is that

$$H^*(BO(2k);R) \xleftarrow{\cong} H^*(BO(2k+1);R) \xrightarrow{\cong} H^*(BSO(2k+1);R)$$

and all are given by

$$R[p_1,\ldots,p_k] .$$

# 37 Steenrod operations

We worked hard to show that mod 2 cohomology takes values not just in graded $\mathbb{F}_2$-vector spaces, but actually in graded commutative $\mathbb{F}_2$-algebras. This additional structure has proven extremely useful. What other natural structure is there on mod 2 cohomology? Both the sum and the cup product are natural operations on two variables. The identity element $1 \in H^0$ is in a sense a natural operation on zero variables (and is the only nonzero natural element in mod 2 cohomology). This invites the question: are there nontrivial natural operations in one variable? Some of course are generated from the product: $x \mapsto x^r$, for example. When $r$ is a power of 2, this is an *additive* operation. We know one other additive operation as well: the *Bockstein*,

$$\beta : H^n(X) \to H^{n+1}(X) .$$

(All our coefficients will be in $\mathbb{F}_2$ in this lecture.) This is obtained as the boundary map in the long exact sequence associated to the short exact sequence $0 \to C_2 \to C_4 \to C_2 \to 0$.

Our goal in this lecture is to establish the following theorem, due to Norman Steenrod (1910–1971, working at Princeton).

**Theorem 37.1.** *For any $n \geq 0$, there is a unique family of additive natural transformations*

$$\mathrm{Sq}^k : H^n \to H^{n+k} , \quad k \geq 0 ,$$

*such that*

$$\mathrm{Sq}^0 x = x , \quad \mathrm{Sq}^k(x) = x^2 \text{ if } k = |x| , \quad \mathrm{Sq}^k x = 0 \text{ if } k > |x| ,$$

*and the "Cartan formula"*

$$\mathrm{Sq}^k(xy) = \sum_{i+j=k} (\mathrm{Sq}^i x)(\mathrm{Sq}^j y)$$

*is satisfied.*

It will transpire that $\mathrm{Sq}^1 = \beta$.

By the Yoneda lemma, natural transformations $H^n \to H^{n+k}$ are classified by $H^{n+k}(K_n)$, where we write

$$K_n = K(\mathbb{F}_2, n)\,.$$

We won't try to compute the whole of $H^*(K_n)$, at least not right away, though eventually it will turn out that the entire cohomology of the mod 2 Eilenberg Mac Lane spaces is generated as an algebra by iterates of the operations we will construct. But at least we can notice right off that

$$H^i(K_n) = 0 \quad \text{for} \quad 0 < i < n$$

and

$$H^n(K_n) = \mathbb{F}_2 \quad \text{for} \quad n > 0$$

by the Hurewicz theorem, so the only nonzero operation on $n$-dimensional classes that lowers degrees is the one sending every $x$ to $1 \in H^0$.

The starting point is the failure of the map cochain cross product

$$S^*(X) \otimes S^*(X) \to S^*(X \times X)$$

– or of *any* natural chain map inducing the cross product in cohomology – to be $C_2$-equivariant. This failure reflects itself geometrically using the following construction.

**Definition 37.2.** The *extended square* of a space $X$ is the balanced product

$$S^\infty \times_{C_2} X^2\,.$$

Here $C_2$ acts antipodally on $S^\infty$, and swaps the factors in $X^2$.

This is the total space of the bundle with fiber $X^2$ associated to the universal principal $C_2$ bundle $S^\infty \downarrow \mathbb{R}P^\infty$. We will study it by means of the Serre spectral sequence.

Actually, it will be important to consider a pointed refinement of this. So suppose given a basepoint $* \in X$. It determines the subset

$$X \vee X \subseteq X \times X$$

consisting of the "axes" in the product. The *pair* $(X^2, X \vee X)$ is equivariant, and determines a bundle pair

$$S^\infty \times_{C_2} (X^2, X \vee X) \downarrow \mathbb{R}P^\infty\,.$$

A point in $S^\infty$ determines a fiber inclusion

$$i : (X^2, X \vee X) \to S^\infty \times_{C_2} (X^2, X \vee X)\,.$$

We'll be working with the cohomology Künneth theorem, so let's restrict ourselves to spaces whose mod 2 cohomology is of finite type. Serre's mod $\mathcal{C}$ theory guarantees that $K_n$ is in this category, and the Künneth theorem guarantees that the category is closed under products.

**Proposition 37.3.** *There is a unique natural transformation*

$$P : \overline{H}^n(X) \to H^{2n}(S^\infty \times_{C_2} (X^2, X \vee X))$$

*such that*

$$i^* P(x) = x^{\otimes 2} \in H^{2n}(X^2, X \vee X)\,.$$

*Proof.* We'll study the associated Serre spectral sequence,

$$H^s(\mathbb{R}P^\infty; H^t(X^2, X \vee X)) \underset{s}{\Longrightarrow} H^{s+t}(S^\infty \times_{C_2} (X^2, X \vee X)).$$

While the chain-level cross product isn't equivariant, the cohomology cross product is: The cross relative product map

$$\overline{H}^*(X) \otimes \overline{H}^*(X) \to H^*(X^2, X \vee X)$$

is equivariant, if we let $C_2$ act by exchanging factors on the left and on the right. This map is an isomorphism if $H^*(X)$ is of finite type, and then the $\mathbb{F}_2[C_2]$-module featuring as coefficients in the spectral sequence can be written as $\overline{H}^*(X)^{\otimes 2}$. It's interesting and not hard to analyze this representation of $C_2$, but we do not need to know about that to construct Steenrod operations. All we need to know is that any $x \in \overline{H}^n(X)$ determines an invariant class $x \otimes x \in \overline{H}^n(X)^{\otimes 2}$.

Now comes the trick: It suffices to consider the universal example, $\iota_n \in \overline{H}^n(K_n)$. Since $\overline{H}^i(K_n) = 0$ for $i < n$, the entire $E_2$ term of

$$H^s(\mathbb{R}P^\infty; H^t(K_n^2, K_n \vee K_n)) \Longrightarrow H^*(S^\infty \times_{C_2} (K_n^2, K_n \vee K_n))$$

lies in vertical dimensions $t \geq 2n$.

So the group

$$E_2^{0,2n} = H^{2n}(K_n^2, K_n \vee K_n) = \langle \iota_n \otimes \iota_n \rangle$$

survives to $E_\infty^{0,2n}$. The element $\iota_n \otimes \iota_n$ lifts to an element of $H^{2n}(S^\infty \times_{C_2} (K_n^2, K_n \vee K_n))$, and this lift is unique because all the lower filtration degrees vanish. This lifted class is $P\iota_n$. By definition (and the edge homomorphism story) it restricts on $(K_n^2, K_n \vee K_n)$ to $\iota_n \otimes \iota_n$. □

The resulting natural transformation $P : H^n(X) \to H^n(S^\infty \times_{C_2} (X^2, X \vee X))$ is the "total square." It's a prime example of a "power operation."

Now we "internalize," by pulling back under the diagonal map. The "commutativity" of the diagonal map becomes important:

$$\Delta : X \to X \times X$$

is equivariant, where $C_2$ acts trivially on $X$ and by swapping the factors in $X \times X$. It induces a map

$$S^\infty \times_{C_2} (X, *) \to S^\infty \times_{C_2} (X^2, X \vee X).$$

But

$$S^\infty \times_{C_2} (X, *) = \mathbb{R}P^\infty \times (X, *)$$

so we have

$$\delta : \mathbb{R}P^\infty \times (X, *) \to S^\infty \times_{C_2} (X^2, X \vee X).$$

Pick $x \in \overline{H}^n(X)$ and consider the pullback $\delta^* P(x)$. By the Künneth theorem,

$$H^*(\mathbb{R}P^\infty \times (X, *)) = H^*(\mathbb{R}P^\infty) \otimes \overline{H}^*(X)$$

so $\delta^* P(x)$ has an expression as a polynomial in the generator $t \in H^1(\mathbb{R}P^\infty)$. The coefficients are the Steenrod squares:

$$\delta^* P(x) = (\mathrm{Sq}^n x) + (\mathrm{Sq}^{n-1} x)t + \cdots + (\mathrm{Sq}^0 x)t^n \quad \mathrm{Sq}^i x \in \overline{H}^{n+i}(X).$$

Since $\overline{H}^i(K_n) = 0$ for $i < n$, there are no natural transformations that decrease degree: so there are no negatively indexed squares; the sum terminates as indicated.

Any operation on $\overline{H}^*$ induces one on $H^*$ by using the isomorphism

$$H^*(X) = \overline{H}^*(X_+)\,.$$

Note that $(X_+)^2 = X^2 \sqcup (X_+ \vee X_+)$ so the total square specializes to a natural transformation

$$P : H^n(X) \to H^{2n}(S^\infty \times_{C_2} X^2)\,.$$

**Proposition 37.4.** $\mathrm{Sq}^n : H^n \to H^{2n}$ *is the squaring map* $x \mapsto x^2$.

*Proof.* This is the coefficient of $1 \in H^0(\mathbb{R}P^\infty)$, so we should pick a basepoint for $\mathbb{R}P^\infty$, and watch the evolution of the class $Px$ in the cohomology of the commutative diagram

$$\begin{array}{ccc} * \times X & \xrightarrow{\Delta} & S^0 \times_{C_2} X^2 = X^2 \\ \downarrow & & \downarrow \\ \mathbb{R}P^\infty \times X & \xrightarrow{\delta} & S^\infty \times_{C_2} X^2 \end{array} \qquad \begin{array}{ccc} 1 \otimes \mathrm{Sq}^n x = x^2 & \leftarrow\!\shortmid & x \otimes x \\ \uparrow & & \uparrow \\ 1 \otimes \mathrm{Sq}^n x + \cdots & \leftarrow\!\shortmid & Px\,. \end{array}$$

□

**Proposition 37.5.** $\mathrm{Sq}^1 = \beta$.

*Proof.* Acting on $H^q$ for $q \geq 1$, both $\mathrm{Sq}^1$ and $\beta$ are nonzero. (Exercise: Provide examples.) We claim that $\dim H^{n+1}(K_n) = 1$ for $n \geq 1$, so the two must coincide. Since $K_1 = \mathbb{R}P^\infty$, we know that case. For the inductive step, use the Serre exact sequence on the fibration sequence

$$K_{n-1} \to PK_n \to K_n\,.$$

□

How about $\mathrm{Sq}^0$? Since $\overline{H}^n(K_n) = \mathbb{F}_2$, there are only two natural transformations $\overline{H}^n \to \overline{H}^n$: the identity and the zero map. The Steenrod operation $\mathrm{Sq}^0$ is one or the other; which is it? In a sense the operations $\mathrm{Sq}^k$ get more sophisticated as $k$ decreases; identifying $\mathrm{Sq}^0$ is tricky. In fact there are many other contexts in which Steenrod operations can be defined, and in a sense the topological context is characterized by $\mathrm{Sq}^0 = 1$. We'll study the simplest case first.

**Proposition 37.6.** $\mathrm{Sq}^0 = 1$ *on* $\overline{H}^1$.

*Proof.* It suffices to come up with a single example of a space with a nonzero class $x \in \overline{H}^1(X)$ such that $\mathrm{Sq}^0 x = x$. Our example will be $S^1$ with the generator $x \in \overline{H}^1(S^1)$.

It suffices to look at the subspace of the extended square in which $S^\infty$ is replaced by $S^1$. Passing to the quotient space of the pair $S^1 \times_{C_2} (S^1 \times S^1, S^1 \vee S^1)$, we arrive at the pointed space

$$\frac{S^1 \times_{C_2} (S^1 \wedge S^1)}{S^1 \times_{C_2} *}$$

in which $C_2$ exchanges the two factors of $S^1$. The smash product may be identified with the one-point compactification of $\mathbb{R}^2$, with $C_2$ acting linearly by permuting the two basis vectors. This representation of $C_2$ is just $1 \oplus \sigma$, the sum of the trivial 1-dimensional representation with the sign representation.

We have the double cover $S^1 \downarrow \mathbb{R}P^1$. This is a principal $C_2$-bundle, and the space we are looking at is exactly the Thom space of the vector bundle over $\mathbb{R}P^1$ associated to this principal $C_2$ bundle and the representation $1 \oplus \sigma$: it is $\mathrm{Th}(\epsilon \oplus \lambda)$ where $\lambda$ is the tautological line bundle over $\mathbb{R}P^1$. Thus we arrive at

$$\frac{S^1 \times_{C_2} (S^1 \wedge S^1)}{S^1 \times_{C_2} *} = \Sigma \mathbb{R}P^2 .$$

The fiber inclusion into the extended square corresponds under this identification with the fiber inclusion in the Thom space. So the nontrivial class in $H^2(\Sigma\mathbb{R}P^2)$ is the Thom class; it restricts to $x \otimes x$ in the fiber, and hence the Thom class is the total square $Px$.

The diagonal inclusion

$$\frac{S^1 \times_{C_2} S^1}{S^1 \times_{C_2} *} \to \frac{S^1 \times_{C_2} (S^1 \wedge S^1)}{S^1 \times_{C_2} *}$$

corresponds to including the fixed point subspace into the representation $1 \oplus \sigma$. This produces a bundle map $\epsilon \to \epsilon \oplus \lambda$ covering the inclusion $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$. We obtain a map of Thom spaces

$$\Sigma \mathbb{R}P^1_+ \to \Sigma \mathbb{R}P^2$$

that (by naturality of the Thom isomorphism) is an isomorphism in dimension 2. This is generated by the class $t \otimes x$, and we conclude that $\mathrm{Sq}^0 x = x$. □

The Cartan formula is quite easy to verify as well, but we won't carry that out here. Notice though that it has an important corollary.

**Proposition 37.7.** *The Steenrod operations are* stable*: For all $n$ and $q$ the diagram*

$$\begin{array}{ccc} \overline{H}^q(X) & \xrightarrow{\mathrm{Sq}^n} & \overline{H}^{q+n}(X) \\ \downarrow \sigma & & \downarrow \sigma \\ \overline{H}^{q+1}(\Sigma X) & \xrightarrow{\mathrm{Sq}^n} & \overline{H}^{q+n+1}(\Sigma X) \end{array}$$

*commutes.*

*Proof.* The suspension isomorphism is induced by the relative cross product

$$\wedge : \overline{H}^1(S^1) \otimes \overline{H}^q(X) \to \overline{H}^{q+1}(\Sigma X) .$$

The Cartan formula together with the fact that $\mathrm{Sq}^0 = 1$ on $\overline{H}^1$ gives the result. □

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**Corollary 37.8.** $\mathrm{Sq}^0$ *is the identity on $\overline{H}^q$ for any $q$.*

*Proof.* We just check this on $\iota_q \in H^q(K_q)$. The map $K_1 \times K_{q-1} \to K_q$ representing the cup product sends $\iota_1 \otimes \iota_{q-1}$ to $\iota_q$, and the result then follows by induction and the Cartan formula. □

**Corollary 37.9.** $\mathrm{Sq}^n : \overline{H}^q \to \overline{H}^{q+n}$ *is additive.*

This is surprising, since the total power operation $P$ is not additive.

*Proof.* Any stable operation $K_q \to K_{q+n}$ is additive: Being stable means that

$$\begin{array}{ccc} K_q & \xrightarrow{\mathrm{Sq}^n} & K_{q+n} \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^k K_{q+k} & \xrightarrow{\Omega^k \mathrm{Sq}^n} & K_{q+k+n} \end{array}$$

commutes up to homotopy. The $H$-space structure of $K_q$ as a loop space is the structure representing the sum in $H^q$, so $Sq^n : K_q \to K_{q+n}$ induces a homomorphism in $[X, -]$. □

The *Steenrod algebra* $A^*$ is the algebra of cohomology operations generated by the Steenrod operations. This is a noncommutative graded $\mathbb{F}_2$-algebra. It is not a free algebra: the Steenrod operations satisfy relations, starting with $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. In fact, all relations among them are determined by two facts:

- $\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0$ and
- The assignment $\mathrm{Sq}^n \mapsto \mathrm{Sq}^{n-1}$ extends to a derivation on $A^*$.

An explicit generating family of relations is given by the *Adem relations*

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_k \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k , \quad i < 2j .$$

(José Adem, 1921–1991, was a student of Steenrod and a founding father of algebraic topology in Mexico.) This relation looks quadratic, and almost is, but fails to be whenever the binomial coefficient with $k = 0$ in the summation is nonzero. If $n$ is not a power of 2, let $j$ be the largest power of 2 less than $n$ and $i = n - j$. Then the binomial coefficient $\binom{j-1}{i}$ is nonzero, so the Adem relation shows that $\mathrm{Sq}^n$ is *decomposable*: a sum of products of positive-dimensional elements. From this we learn:

**Proposition 37.10** (Adem)**.** *$A^*$ is generated by* $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8, \ldots$.

This leads to information about the "Hopf invariant." Among its many interpretations, the Hopf invariant asks how far the sequence of 3-cell complexes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$, can be extended. The "octonions" $\mathbb{O}$ provide us with one more, $\mathbb{O}P^2$. Adem's theorem puts a first restriction on such spaces:

**Corollary 37.11.** *Suppose there is a space $X$ such that $H^*(X) = \mathbb{F}_2[x]/x^3$. Then $|x|$ is a power of 2.*

*Proof.* Let $n = |x|$. Then $\mathrm{Sq}^n x = x^2 \neq 0$. But if $n$ is not a power of 2, this operation factors through groups between dimension $n$ and $2n$. □

This theorem was improved by Frank Adams to: $|x| = 1, 2, 4$ or $8$; there are no examples beyond the classical ones. (John Frank Adams (1930–1989) was a key figure in the development of twentieth century homotopy theory, Lowndean Professor at Cambridge University.)

# 38 Cobordism

René Thom [41] (1923–2002), IHES, discovered how to use all this machinery to give a classification of closed manifolds, which, while crude, is valid in all dimensions. His equivalence relation was *cobordism* (or "bordism" – opinions vary):

**Definition 38.1.** Let $M$ and $N$ be two closed smooth $n$-manifolds. A *cobordism* between them is an $(n+1)$-manifold-with-boundary $W$ together with a diffeomorphism

$$\partial W \cong M \sqcup N\,.$$

If there is a cobordism, $M$ and $N$ are said to be "cobordant."

If $M$ and $N$ are diffeomorphic, we may use $W = M \times I$ along with the diffeomorphism at one end to see that they are cobordant. Cobordism is an equivalence relation on the class of closed $n$-manifolds. Disjoint union endows the set (why "set"?)

$$\mathcal{N}_n = \Omega_n^O$$

of cobordism classes of $n$-manifolds with the structure of a commutative monoid. In fact it is a vector space over $\mathbb{F}_2$, since the same cylinder can be regarded as a null-bordism of $M \sqcup M$. The product of manifold actually renders the collection of bordism groups a graded commutative algebra. Thom proved:

**Theorem 38.2** (Thom)**.** $\mathcal{N}_* = \mathbb{F}_2[x_i : i+1$ *is positive and not a power of* $2]$ *, where* $|x_i| = i$.

We will sketch his proof of this amazing classification theorem over the next few lectures. (Thom actually only proved the additive statement. Bob Stong's notes [38] provide an excellent secondary source.)

Thom also addressed a question formulated by Norman Steenrod – but this question must have been in Poincaré's mind much earlier. There are two competing notions of an $n$-cycle: the singular one we have been using (or the equivalent but even more combinatorial version involving simplicial complexes), and the notion of the fundamental cycle of a closed $n$-manifold. Are they equivalent? Here's Steenrod's formulation of this question. Given an $n$-dimensional mod 2 homology class $x$ in a space $X$, is there a closed $n$-manifold $M$ and a continuous map $f : M \to X$ such that $f_*[M] = x$?

This question has an obvious integral variant as well, in which we demand that the manifold $M$ is oriented.

**Theorem 38.3** (Thom [41])**.** *The answer to these questions are: "Yes" in the unoriented case and "No" in the oriented case.*

## The Pontryagin-Thom collapse

A smooth map $f : M \to N$ of manifolds is an *immersion* if it induces a monomorphism on all tangent spaces. One then has an embedding of vector bundles over $M$, $df : \tau_M \hookrightarrow f^*\tau_N$. The quotient bundle is the *normal bundle* of $f$, $\nu_f$. If we equip $\tau_N$ with a metric, we receive a metric on $f^*\tau_N$ and can identify $\nu_f$ with the orthogonal complement of $\tau_M$ in $f^*\tau_N$:

$$\tau_M \oplus \nu_f \cong f^*\tau_N\,.$$

Suppose that $M$ is compact. An *embedding* $f : M \to N$ is an injective immersion: an immersion without double points. In that case, the *tubular neighborhood theorem* (see [4, p. 93], for example) asserts that the subspace $f(M) \subseteq N$ admits a "regular" neighborhood that is equipped with a diffeomorphism rel $M$ to the normal bundle $\nu_f$. This regular neighborhood is moreover unique up to diffeomorphism rel $M$. In view of this identification we will denote the regular neighborhood by $E(\nu)$.

This observation provides a contravariant relationship between $M$ and $N$: collapse the complement of $E(\nu)$ to a point. This provides a map

$$c : N_+ \to \mathrm{Th}(\nu)$$

from the one-point compactification of $N$ to the Thom space of the normal bundle. This is the *Pontryagin-Thom collapse*. It's a special case of the fact that one-point compactification provides a *contravariant* functor on the category of locally compact Hausdorff spaces and open inclusions.

When $N = \mathbb{R}^{n+k}$, this construction associates to an embedded $n$-manifold $j : M \hookrightarrow \mathbb{R}^{n+k}$ a map $S^{n+k} \to \mathrm{Th}(\nu_j)$. If we vary the embedding through an isotopy (a smooth homotopy through embeddings) and vary the tubular neighborhood, the resulting maps vary through a homotopy.

Now comes Thom's observation: the normal bundle is classified by a map $M \to BO(k)$, which induces a map on the level of Thom spaces. By composing, we get a map

$$S^{n+k} \to \mathrm{Th}(\nu_j) \to \mathrm{Th}(\xi_k) = MO(k)\,.$$

This provides a map from the set of isotopy classes of embeddings of $n$-manifolds into $\mathbb{R}^{n+k}$ to the homotopy group $\pi_{n+k}(MO(k))$. Separated disjoint unions get sent to the sum in the homotopy group. The empty manifold gets sent to zero.

But homotopy corresponds to a still broader equivalence relation on embedded $n$-manifolds. Given $M_0$ and $M_1$, both embedded in $\mathbb{R}^{n+k}$, an *ambient cobordism* between them is a manifold with boundary, $W$, embedded in $\mathbb{R}^{n+k} \times I$, meeting $\mathbb{R}^{n+k} \times 0$ and $\mathbb{R}^{n+k} \times 1$ transversely in $M_0$ (along $\mathbb{R}^{n+k} \times 0$) and $M_1$ (along $\mathbb{R}^{n+k} \times 1$). Isotopies provide cobordisms, but the cobordism could have some more complicated topology as well, and the ends of a cobordism do not have to be even homotopy equivalent. It's not hard to see that cobordisms produce homotopies. Here's the geometric content of Thom's work.

**Theorem 38.4** (Thom)**.** *The Pontryagin-Thom collapse map from the set of ambient cobordism classes of closed $n$-manifolds in $\mathbb{R}^{n+k}$ to the corresponding homotopy class in $\pi_{n+k}(MO(k))$ is bijective.*

For example, $MO(1) = \mathbb{R}P^\infty$, so $\pi_2(MO(1)) = 0$: a union of $i$ circles embedded in $\mathbb{R}^2$ can be written as the boundary of a 2-sphere with $i$ discs removed.

The inverse map is just as interesting. Start with a map

$$f : S^{n+k} \to MO(k)\,.$$

Compress it through an approximation,

$$g : S^{n+k} \to \mathrm{Th}(\xi_{q,k} \downarrow \mathrm{Gr}_k(\mathbb{R}^q))\,.$$

Approximate this by a nearby (and hence homotopic) map that is smooth on the pre-image of $E(\xi_{q,k})$, and deform it further so that it meets the image $Z$ of the zero section transversely. Then the implicit function theorem guarantees that the preimage $g^{-1}(Z)$ is a submanifold $M \hookrightarrow S^{n+k}$. The zero section has codimension $k$ in $E(\xi_{q,k})$, so $M$ is an $n$-manifold.

This construction is pretty clearly inverse to the Pontryagin-Thom collapse. The whole story generalizes to allow structure on the normal bundle: for example an orientation or a complex structure or a trivialization. The key observation is that the normal bundle of the zero section in the Thom space of an appropriate manifold approximation of the relevant universal bundle can be identified with the restriction of the universal bundle and so inherits the same structure. The relevant homotopy groups are then $\pi_{n+k}(MSO(k))$ or $\pi_{n+k}(MU(k/2))$ in the first two cases. Giving a trivialization of a vector bundle is the same thing as giving an isomorphism with the pullback of a bundle over a point, so we can take a point as the corresponding classifying space. The Thom space is a sphere; so in that case the relevant homotopy group is $\pi_{n+k}(S^k)$. This gives a spectacular interpretation of the homotopy groups of spheres. It is the case Pontryagin considered.

## Stabilization

Now it is definitely interesting to consider embedded manifolds, but perhaps abstract manifolds, without a chosen embedding, are even more interesting, or at least simpler. Whitney proved that any closed manifold embeds in Euclidean space of twice its dimension, and if you allow the ambient space to be of even higher dimension you find that any two embeddings are isotopic. Similarly, in high codimension the cobordisms become unconstrained.

Passing from an embedding in $\mathbb{R}^{n+k}$ to an embedding in $\mathbb{R}^{n+k+1}$ replaces the normal bundle $\nu$ with $\nu \oplus \epsilon$. Correspondingly, the map $BO(k) \to BO(k+1)$ classifies $\xi_k \oplus \epsilon$. This gives us maps

$$\Sigma MO(k) \to MO(k+1)$$

for each $k \geq 1$, and hence maps

$$\pi_{n+k}(MO(k)) \to \pi_{n+k+1}(MO(k+1)) \to \pi_{n+k+2}(MO(k+2)) \to \cdots$$

that correspond to considering manifolds embedded in higher and higher dimension. We also get maps in homology,

$$\overline{H}_{n+k}(MO(k)) \to \overline{H}_{n+k+1}(MO(k+1)) \to \overline{H}_{n+k+2}(MO(k+2)) \to \cdots$$

This is a beautiful and motivating example of a (topological!) *spectrum*: A sequence of pointed spaces $E_k$ together with maps $\Sigma E_k \to E_{k+1}$. The direct limit

$$\pi_n(E) = \lim_{k\to\infty} \pi_{n+k}(E_k)$$

is the $n$th *homotopy group* of the spectrum $E$. Similarly we can define the homology of the spectrum $E$ as

$$H_i(E) = \lim_{k\to\infty} \overline{H}_{n+k}(E_k)\,.$$

Spectra are by default "pointed"; there's no "unreduced" homology of a spectrum.

We have already seen a number of other spectra! For example, the *Eilenberg Mac Lane spectrum* $HA$ for the abelian group $A$ has $K(A,n)$ as its $n$th space, and the map $\Sigma K(A,n) \to K(A,n+1)$ that classifies the suspension of the fundamental class – the adjoint of the equivalence $K(A,n) \to \Omega K(A,n+1)$.

Spectra are the central objects of study in stable homotopy theory. Here's a tiny part of that theory. As an endofunctor of the stable homotopy category, suspension is an equivalence. It is a consequence of the definition of homotopy equivalence for spectra that the following two proposed definitions of the suspension of a spectrum $E$ are equivalent.

- $(\Sigma E)_n = \Sigma E_n$, and the bonding maps are the suspensions of the bonding maps in $E$;

- $(\Sigma E)_n = E_{n+1}$, and the bonding maps are the same.

So for example $\Sigma HA$ is equivalently given by

$$\Sigma K(A,0), \Sigma K(A,1), \ldots \quad \text{and} \quad K(A,1), K(A,2), \cdots .$$

The second definition of suspension is clearly a categorical equivalence on the category of spectra.

The spectrum built from Thom spaces as above is the *unoriented Thom spectrum*, and is denoted simply $MO$. The space $MO(k)$ is $(k-1)$-connected, so the Freudenthal suspension theorem assures us that the direct limit defining $\pi_n(MO)$ is achieved. We also have Thom spectra $MSO$ and $MU$; the Thom spectrum corresponding to framed manifolds is the *sphere spectrum* $S$, with $n$th space $S^n$.

The ambient cobordism theorem stabilizes to give:

**Theorem 38.5** (Thom)**.** *The Pontryagin-Thom construction gives an isomorphism from the group of cobordism classes of closed $n$-manifolds to $\pi_n(MO)$:*

$$\mathcal{N}_n \xrightarrow{\cong} \pi_n(MO) .$$

So Thom's classification theorem amounts to computing the homotopy groups of the Thom spectrum $MO$.

### Characteristic numbers

To compute these homotopy groups we need a way to distinguish cobordism classes from each other: We need a supply of "cobordism invariants." Characteristic classes afford such invariants.

Let $M$ be an $n$-manifold. Embed it in some Euclidean space, $M \hookrightarrow \mathbb{R}^{n+k}$, and denote the normal bundle of the embedding by $\nu$. Its mod 2 characteristic classes are polynomials in the Stiefel-Whitney classes; there are lots of them. The ones that happen to lie in $H^n(M)$ can be paired against the fundamental class $[M]$. The resulting elements of $\mathbb{F}_2$ are *characteristic numbers.*

**Lemma 38.6.** *Characteristic numbers are cobordism invariants.*

*Proof.* We have to show that if $M = \partial N$ then

$$\langle w(\nu), [M] \rangle = 0$$

for any $w \in H^n(BO)$. The class $[M]$ is the boundary of the relative fundamental class $[N,M] \in H^{n+1}(N,M)$, so using the adjointness of the boundary and coboundary maps

$$\langle w(\nu), [M] \rangle = \langle \delta w(\nu), [N,M] \rangle .$$

We claim that $\delta w(\nu) = 0$, and we will show that by exhibiting a class in $H^n(N)$ that restricts to $w(\nu)$. By increasing the codimension if necessary, we can assume that the bounding manifold $W$ embeds in $\mathbb{R}^{n+k} \times [0,\infty)$, meeting $\mathbb{R}^{n+k} \times 0$ transversely in $M$. So the normal bundle $\nu$ extends the normal bundle $\nu_N$ of $N \hookrightarrow \mathbb{R}^{n+k} \times [0,\infty)$, and $w(\nu) = w(i^*\nu_N) = i^* w(\nu_N)$ (where $i : M \hookrightarrow N$ is the inclusion of the boundary). □

Putting all the characteristic numbers in play at once, we get the "characteristic number map"

$$\mathcal{N}_n \to \operatorname{Hom}(H^n(BO), \mathbb{F}_2) = H_n(BO) .$$

We'll reinterpret this map in terms of the Thom spectrum $MO$.

Let $\xi$ be a real $n$-plane bundle over a space $B$. The cohomology Thom isomorphism relied on the pairing

$$\mathrm{Th}(\xi) \to B_+ \wedge \mathrm{Th}(\xi)\,,$$

and was given by pairing with the Thom class $U \in H^n(\mathrm{Th}(\xi))$. In homology, this pairing produces the top row in

$$\begin{array}{ccc} \overline{H}_{*+n}(\mathrm{Th}(\xi)) & \longrightarrow & H_*(B) \otimes \overline{H}_n(\mathrm{Th}(\xi)) \\ & \searrow^{\cong} & \downarrow {\scriptstyle 1\otimes\langle U,-\rangle} \\ & & H_*(B) \end{array}$$

The vertical map is defined using the Kronecker pairing with the Thom class. The diagonal map is the homology Thom isomorphism.

In the universal case we have isomorphisms

$$\overline{H}_{*+n}(MO(n)) \xrightarrow{\cong} H_n(BO(n))\,.$$

These maps are compatible with stabilization and give the Thom isomorphism

$$\Phi : H_*(MO) \xrightarrow{\cong} H_*(BO)\,.$$

These constructions fit together in the commutative diagram:

$$\begin{array}{ccccc} \mathcal{N}_n & \xrightarrow{\alpha} & \mathrm{Hom}(H^n(BO;\mathbb{F}_2),\mathbb{F}_2) & & \\ & & & \nwarrow^{\beta}_{\cong} & \\ \downarrow & & & & H_n(BO;\mathbb{F}_2) \\ & & & \swarrow^{\Phi}_{\cong} & \\ \pi_n(MO) & \xrightarrow{h} & H_n(MO;\mathbb{F}_2)\,. & & \end{array}$$

Thom proved that the mod 2 Hurewicz homomorphism $h$ is a monomorphism. As a corollary:

**Corollary 38.7.** *If the closed $n$-manifolds $M$ and $N$ have the same Stiefel-Whitney numbers, then they are cobordant.*

This uses algebraic topology to guarantee a very geometric outcome! For example, if all the Stiefel-Whitney numbers vanish then the manifold is *null-bordant*: it is the boundary of some $(n+1)$-manifold-with-boundary.

Thom's basic homotopy-theoretic theorem is this:

**Theorem 38.8** (Thom)**.** *The spectrum $MO$ is a product of suspensions of the mod 2 Eilenberg Mac Lane spectrum.*

This implies a positive solution to Steenrod's question. A convenient way to explain this is via an observation of Michael Atiyah [2]. Let $X$ be any space (a "background," in physics parlance), and consider the set of continuous maps from closed $n$-manifolds into $X$, modulo the equivalence relation given by cobordism of manifolds together with extension of the maps. This is an abelian group depending covariantly on $X$,

$$X \mapsto \Omega^O_n(X)$$

Atiyah showed that it is a generalized homology theory. Its "coefficients" are

$$\Omega_n^O(*) = \mathcal{N}_n\,.$$

There is a natural map, the "Thom reduction,"

$$\Omega_n^O(X) \to H_n(X;\mathbb{F}_2)$$

given by sending $f : M \to X$ to $f_*([M]) \in H_n(X;\mathbb{F}_2)$. Steenrod's question asks whether this map is surjective.

Generalized homology theories are "represented" by spectra. Given a spectrum $E$ and a pointed space $Y$, one can form the "smash product" spectrum $E \wedge Y$ with

$$(E \wedge Y)_n = E_n \wedge Y$$

and the obvious bonding maps.

**Theorem 38.9** (George Whitehead and Edgar Brown)**.** *Given any spectrum $E$, the functors*

$$E_* : X \mapsto \pi_n(E \wedge X_+)$$

*constitute a generalized homology theory, and any generalized homology theory admits such a representation.*

In particular

$$\Omega_n^O(X) = \pi_n(MO \wedge X_+) \quad \text{and} \quad H_n(X;\mathbb{F}_2) = \pi_n(H\mathbb{F}_2 \wedge X_+)$$

so the fact that there is a section of the Thom class $U : MO \to H\mathbb{F}_2$ (given by including the bottom factor into the product) implies a positive answer to Steenrod's question.

# 39 Hopf algebras

## Product structure

There is more structure to exploit in our study of the bordism groups. The product of a closed $m$-manifold $M$ and a closed $n$-manifold $N$ is a closed $(m+n)$-manifold. This is what gives $\Omega_*^O = \Omega_*^O(*)$ its structure as a commutative graded ring. To pass this through the Pontryagin-Thom collapse, notice that $M \times N$ embeds into the product of ambient Euclidean spaces, and the resulting normal bundle is the product of the two normal bundles. The universal case of a product of $m$-plane and $n$-plane bundles is represented by a map

$$BO(m) \times BO(n) \to BO(m+n)$$

which is covered by the bundle map $\xi_m \times \xi_n \to \xi_{m+n}$ and hence induces a map on the level of Thom spaces:

$$MO(m) \wedge MO(n) \to MO(m+n)\,.$$

These maps render $MO$ a "ring spectrum," making $\pi_*(MO)$ a graded ring, and the map

$$\Omega_*^O \to \pi_*(MO)$$

is a ring isomorphism. Equally, $H_*(MO)$ is a graded ring and the Hurewicz map is a ring homomorphism. The homology Thom isomorphism is also multiplicative: The space $BO$ has a commutative $H$-space structure derived from Whitney sum, and the map $\Phi : H_*(BO) \to H_*(MO)$ is an isomorphism of graded rings.

## Hopf algebras

With a field for coefficients, the Künneth theorem delivers for any space $X$ a map

$$\Delta : H_*(X) \to H_*(X \times X) \xleftarrow{\cong} H_*(X) \otimes H_*(X)$$

(all tensors over the coefficient field $k$) variously termed a "coproduct," "comultiplication," or "diagonal." The unique map $X \to *$ gives us a "counit" $H_*(X) \to k$.

**Definition 39.1.** A *$k$-coalgebra* is an $k$-module $A$ together with $k$-module maps $\epsilon : A \to k$ and $\Delta : A \to A \otimes A$ that are unital and associative:

$$\begin{array}{ccccc} & & A & & \\ & {}^{1}\swarrow & \downarrow{\scriptstyle\Delta} & \searrow^{1} & \\ A & \xleftarrow{1\otimes\epsilon} & A \otimes A & \xrightarrow{\epsilon\otimes 1} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes 1} \\ A \otimes A & \xrightarrow{1\otimes\Delta} & A \otimes A \otimes A\,. \end{array}$$

It is *commutative* if also

$$\begin{array}{ccc} & A & \\ {}^{\Delta}\swarrow & & \searrow^{\Delta} \\ A \otimes A & \xrightarrow{T} & A \otimes A \end{array}$$

commutes.

This makes sense in the graded context as well, when the swap map $T$ should contribute its usual sign. In that case we say that $A$ is *connected* if $A_i = 0$ for $i < 0$ and $\epsilon : A \to k$ is an isomorphism in dimension 0.

The diagonal in $H_*(X)$ is dual to the cup product: the universal coefficient isomorphism

$$\mathrm{Hom}(H_*(X), k) \cong H^*(X)$$

sends the diagonal to the cup product (and $\epsilon$ to the unit map $k \to H^*(X; k)$).

If $X$ is an $H$-space, the product induces the "Pontryagin product" $\mu : H_*(X) \otimes H_*(X) \to H_*(X)$. Since the product and the basepoint inclusion $* \to X$ are maps of spaces, they are maps of coalgebras. We have to say what the coalgebra structure is on a tensor product of coalgebras, say $A$ and $B$: define

$$\Delta_{A\otimes B} : A \otimes B \xrightarrow{\Delta\otimes\Delta} (A \otimes A) \otimes (B \otimes B) \xrightarrow{1\otimes T\otimes 1} (A \otimes B) \otimes (A \otimes B)$$

and

$$\epsilon_{A\otimes B} : A \otimes B \xrightarrow{\epsilon\otimes\epsilon} k \otimes k = k\,.$$

We have described the structure of a *bialgebra*: an associative multiplication with unit and an associative comultiplication with counit on the same (possibly graded) vector space, that are compatible in the sense that the unit and multiplication are coalgebra maps, or, equivalently, that the counit and comultiplication are algebra maps.

If the $H$-space $X$ has an "inverse" – a map $x \mapsto x^{-1}$ making it into a group in the homotopy category – then $A = H_*(X)$ becomes a *Hopf algebra*: there is a map $\chi : A \to A$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{1\otimes\chi} & H \otimes A & \xrightarrow{\mu} & A \\ & {}_{\epsilon}\searrow & & & & \nearrow_{\eta} & \\ & & & A & & & \end{array}$$

commutes. This "canonical anti-automorphism" $\chi$ exists uniquely if $A$ is a connected graded bialgebra.

An important and motivating example of an ungraded Hopf algebra is given by the group algebra of a group $G$: $k[G]$ admits the diagonal determined by $\Delta g = g \otimes g$ for $g \in G$. The anti-automorphism is induced by the map $g \mapsto g^{-1}$. Indeed, a Hopf algebra with commutative diagonal is just a group object in the category of commutative coalgebras.

The $k$-linear dual of a $k$-coalgebra is a $k$-algebra. If a Hopf algebra is of finite type, its dual is again a Hopf algebra. So if $X$ is an $H$-space of finite type then $H^*(X)$ is also a Hopf algebra; the coproduct comes from the multiplication in $X$. It's a good exercise to go through our list of $H$-spaces and understand the Hopf algebra structure on their homology and cohomology. Here's an example, with coefficients in $\mathbb{F}_2$.

**Proposition 39.2.** *Whitney sum renders $BO$ a commutative $H$-space, and the map $BO(1) \to BO$ sends the vector space generators of $\overline{H}_*(BO(1))$ to polynomial generators $a_i$:*

$$H_*(BO) = \mathbb{F}_2[a_1, a_2, \ldots] .$$

Thus $H_*(BO)$ is "bipolynomial": both homology and cohomology are polynomial algebras.

The diagonal puts strong restrictions on the algebra structure of a Hopf algebra.

**Proposition 39.3** (Hopf and Leray)**.** *Let $A$ be a graded connected Hopf algebra of finite type over a field of characteristic zero, and suppose the product is commutative. Then $A$ is a free commutative graded algebra.*

This means that $A$ is a tensor product of a polynomial algebra on even generators and an exterior algebra on odd generators.

**Corollary 39.4** (Hopf)**.** *The rational cohomology of any connected Lie group is an exterior algebra on odd generators.*

Here's an analogue in finite characteristic.

**Proposition 39.5** (Borel)**.** *Let $A$ be a graded connected Hopf algebra of finite type over a perfect field of characteristic $p$, and suppose that the product is commutative. If $p$ is odd, $A$ is an exterior algebra on odd generators tensored with a polynomial algebra on even generators modulo the ideal generated by $p^k$th powers of some of those generators. If $p = 2$, it is a polynomial algebra modulo $2^k$th powers of some generators.*

## The Steenrod algebra and its dual

Given two modules $M$ and $N$ over a Hopf algebra $A$, their tensor product over $k$ has a canonical structure of module over $A$ again:

$$A \otimes M \otimes N \xrightarrow{\Delta \otimes 1 \otimes 1} A \otimes A \otimes (M \otimes N) \xrightarrow{1 \otimes T \otimes 1} (A \otimes M) \otimes (A \otimes N) \xrightarrow{\varphi \otimes \varphi} M \otimes N$$

When $A = k[G]$, this is the familiar diagonal tensor product of representations.

John Milnor [25] made the observation that the Cartan formula may be formulated in terms of a Hopf algebra structure on the Steenrod algebra itself:

**Proposition 39.6.** *The association*

$$\Delta : \mathrm{Sq}^k \to \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$$

*extends to an algebra map, and provides the (commutative!) coproduct in a Hopf algebra structure on the Steenrod algebra $A^*$.*

The Cartan formula then merely asserts that the cup product $H^*(X) \otimes H^*(X) \to H^*(X)$ is a map of $A^*$-modules.

This is pleasant, but much more striking is the insight this gives you into the structure of the Steenrod algebra. Write $A_*$ for the Hopf algebra dual to $A^*$.

**Proposition 39.7.** *There exist elements $\zeta_i \in A_{2^i-1}$ such that*

$$A_* = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$$

*and (with $\zeta_0 = 1$)*

$$\Delta \zeta_k = \sum_{i+j=k} \zeta_i^{2^j} \otimes \zeta_j \, .$$

This is equivalent to the Adem relations, but it's much easier to remember!

## Lagrange and Thom

**Theorem 39.8.** *$H^*(MO)$ is free as module over the Steenrod algebra $A^*$.*

Thom gave a fairly elaborate combinatorial proof of this theorem, writing down a basis. It turns out that a little bit of Hopf algebra technology makes this a lot simpler (or at least more believable).

**Lemma 39.9** ("Lagrange"; see e.g. [38], p. 94)**.** *Let $A$ be a connected Hopf algebra and $C$ a connected coalgebra with compatible $A$-module structure (so that the counit and diagonal are $A$-module maps). Let $u \in C^0$ be such that $\epsilon u = 1$. If $Au$ is free, then $C$ is free as $A$-module.*

The reference to Lagrange is this: A common application of this lemma is to take $C$ to be a Hopf algebra containing $A$ as a subalgebra. The result is that $C$ is automatically free as an $A$-module. This is analogous to an observation attributed to Lagrange: If $G$ is a group and $H < G$ a subgroup then the translation action of $H$ on $G$ is free.

We will apply it with $A = A^*$ and $C = H^*(MO)$. Then $H^0(MO)$ is generated by the Thom class $U$, so what we have to do is to check that $A^*$ acts freely on the Thom class.

This is proved using the following amazing observation of Thom's:

**Proposition 39.10** ([41])**.** *Let $\xi$ be a vector bundle over $B$, with Thom space* $\mathrm{Th}(\xi)$*. Then*

$$\mathrm{Sq}^i U = w_i \cup U \, .$$

This provides a definition of the Stiefel-Whitney classes that only uses the spherical fibration determined by the vector bundle, and indeed one that makes sense for any spherical fibration. It's quite easy to prove that these classes satisfy the axioms.

**Exercise 39.11.** Let $M$ be a closed smooth $n$-manifold. By Poincaré duality, there is for each $k$ a unique class $v_k \in H^k(M)$ such that $\langle v_k x, [M] \rangle = \langle \mathrm{Sq}^k x, [M] \rangle$ for all $x \in H^{n-k}(M)$. These are the "Wu classes" of the manifold. Show that $\mathrm{Sq} v = w(\tau_M)$. The tangential Stiefel-Whitney classes are therefore homotopy invariants of the manifold. Show that the normal Stiefel-Whitney classes are as well, and conclude that if two closed manifolds are homotopy equivalent then they are cobordant.

### Conclusion

Stably, cohomology is represented by the Eilenberg Mac Lane spectrum. Pick a basis $B$ for $H^*(MO)$ as an $A^*$-module. Each element $b \in B$ determines a homotopy class $MO \to \Sigma^{|b|} H\mathbb{F}_2$. Assembling them gives a map

$$MO \to \prod_{b \in B} \Sigma^{|b|} H\mathbb{F}_2$$

that is an isomorphism in mod 2 cohomology. Since the homotopy of $MO$ is all 2-torsion, this map is actually weak equivalence.

The Eilenberg Mac Lane spectrum $H\mathbb{F}_2$ is a commutative ring spectrum as well; the ring structure represents the cup product in cohomology. Its homology is thus a graded commutative algebra, namely the dual of the Steenrod algebra (which is the *co*homology of $H\mathbb{F}_2$!). We can now estimate the size of $\pi_*(MO)$: Each basis element produces a suspended copy of $A_*$ in $H_*(MO) = \mathbb{F}_2[a_1, a_2, \ldots]$. It looks like the Milnor generators, $\zeta_i \in A_{2^i-1}$ account for some of the $a_i$'s. The rest must come from the homotopy. Some further argumentation leads to the conclusion that

$$\pi_*(MO) = \mathbb{F}_2[x_i : i+1 \text{ is not a power of } 2\,] .$$

## 40 Applications of cobordism

### Oriented cobordism

The Pontryagin-Thom collapse/transversality story is very general, and provides for example an isomorphism

$$\Omega_*^{SO} \cong \pi_*(MSO) .$$

The oriented bordism groups were computed completely by C.T.C. Wall. All torsion is killed by 2. The first few groups are

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\Omega_n^{SO}$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |

Wall's computation is involved, but at least it's quite easy to determine $\pi_*(MSO) \otimes \mathbb{Q}$, by virtue of a general observation.

**Proposition 40.1.** *For any spectrum $E$, the rational Hurewicz map*

$$\pi_*(E) \otimes \mathbb{Q} \to H_*(E; \mathbb{Q})$$

*is an isomorphism.*

There are many ways to see this. For example, up to weak equivalence we may build up a spectrum by attaching cells. Both $\pi_*^s$ and $H_*$ are generalized homology theories; they send cofiber sequences to long exact sequence. So it's enough to show that the map is an isomorphism for the case of the sphere spectrum, where it follows from Serre's calculation of the rational homotopy of spheres.

So we have the commutative diagram of algebra isomorphisms

$$\begin{array}{ccccc}
\Omega_*^{SO} \otimes \mathbb{Q} & \longrightarrow & & & \mathrm{Hom}(H^*(BSO; \mathbb{Q}), \mathbb{Q}) \\
\downarrow & & & & \uparrow \\
\pi_*(MSO) \otimes \mathbb{Q} & \longrightarrow & H_*(MSO; \mathbb{Q}) & \longleftarrow & H_*(BSO; \mathbb{Q})
\end{array}$$

where the top arrow is the characteristic number map sending $[M]$ to $(p \mapsto \langle p(\nu), [M]\rangle)$. This already says something important: The rational Pontryagin numbers of a manifold determine is position in the rational oriented bordism ring. If they all vanish on a manifold $M$, some multiple of $M$ bounds an oriented manifold-with-boundary.

Again, $BSO$ is a commutative $H$-space, so $H_*(BSO;\mathbb{Q})$ is a $\mathbb{Q}$-Hopf algebra, and so by the Hopf-Leray theorem it is a polynomial algebra. Since $H^*(BSO;\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots]$, we find that the homology is also a polynomial algebra on generators of dimension $4k$. An analysis of the characteristic numbers of projective spaces shows that we may take the classes of the even complex projective spaces as the polynomial generators:

$$\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[[\mathbf{C}P^2], [\mathbf{C}P^4], \ldots]\,.$$

## Steenrod operations on the Thom class

When Thom tried to move beyond this rational calculation, and follow his analysis of the homotopy type of $MO$, he ran into trouble at odd primes. There are odd primary Steenrod operations, constructed in the same way as the squares were. (A nice reference for this is [12].) They take the form

$$P^i : H^n(X;\mathbb{F}_p) \to H^{n+2(p-1)i}(X;\mathbb{F}_p)\,.$$

Now $P^0x = x$, $P^nx = x^p$ if $|x| = 2n$, $P^nx = 0$ if $|x| < 2n$. There is also the Bockstein operation $\beta : H^n(X;\mathbb{F}_p) \to H^{n+1}(X;\mathbb{F}_p)$. These operations generate all the additive operations on mod $p$ cohomology. The dual of $A^*$, for $p$ odd, has the form [25]

$$A_* = E[\tau_0, \tau_1, \ldots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots]\,, \quad |\tau_i| = 2p^i - 1\,, \quad |\xi_i| = 2p^i - 2\,.$$

Now $H^1(BSO) = 0$ (we've killed $w_1$!), so $H^1(MSO) = 0$ as well; the Thom class $U \in H^0(MSO)$ is killed by the Bockstein. It turns out that at $p = 2$, $\beta = \mathrm{Sq}^1$ generates the annihilator ideal of $U$. This isn't so bad, since in fact

$$H^*(H\mathbb{Z};\mathbb{F}_2) = A^*/A^*\mathrm{Sq}^1$$

and indeed $MSO_{(2)}$ splits as a product of Eilenberg Mac Lane spectra (but now not just $H\mathbb{F}_2$'s but also $H\mathbb{Z}_{(2)}$'s).

But at an odd prime the situation is worse; the annihilator of $U \in H^0(MSO;\mathbb{F}_p)$ is the left ideal generated by $\beta P^i$ for all $i$. This implies, for example, that $\beta P^1$ kills the Thom class of the normal bundle for any embedding of an oriented manifold into Euclidean space. The Thom spectrum $MSO$ does not split as a product of Eilenberg Mac Lane spectra at an odd prime.

## Duality

To see how this behavior of Steenrod operations on the Thom class leads to Thom's counterexample to the oriented form of Steenrod's question, we have to explain something about duality in homotopy theory. One of the motivations for the development of the stable homotopy category was a desire to make this story smooth. We will be brief, however.

Any finite complex $K$ may be embedded into some finite dimensional Euclidean space $\mathbb{R}^m$. It can be arranged that the complement has a finite subcomplex $L$ as a deformation retract. Alexander duality then gives us an isomorphism

$$\alpha : H_{m-q}(K) \cong \widetilde{H}^{q-1}(L)$$

for any $q$.

A homotopy-theoretic duality underlies this homological duality: $L$ (or an appropriate desuspension of it in the stable homotopy category) is the "Spanier-Whitehead dual" of $K_+$. This geometry implies that with mod $p$ coefficients this isomorphism commutes with the action of Steenrod operations. To make sense of this, use the universal coefficient theorem to reexpress homology as the linear dual of cohomology:

$$H_{m-q}(K) = H^{m-q}(K)^\vee .$$

This imposes a "contragredient" *right* action of $A^*$ on homology, with $\theta \in A^r$ acting in such a way that

$$\langle x, c\theta \rangle = \langle \theta x, c \rangle .$$

The isomorphism $\alpha$ demands a *left* action of $A^*$, which is achieved by acting in homology by $\overline{\theta}$ where $\theta \mapsto \overline{\theta}$ is the Hopf anti-automorphism. The duality isomorphism is compatible with this action; that is, for $c \in H_{m-q}(K)$,

$$\theta(\alpha c) = \alpha(c\overline{\theta}) .$$

Now suppose that $M \hookrightarrow \mathbb{R}^{n+k}$ is an embedding of a closed manifold, with normal bundle $\nu$. Let $N$ be the closure of a regular neighborhood of $M$; it may be identified with $\mathbb{D}(\nu)$.

The complement $\mathbb{R}^{n+k} - E(\nu)$ is our finite complex $L$. Here's an important point: we have equivalent cofiber sequences

$$\begin{array}{ccccccc}
\mathbb{R}^{n+k} - E(\nu) & \longrightarrow & \mathbb{R}^{n+k} & \longrightarrow & \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - E(\nu)) & \cong & \mathbb{D}(\nu)/\mathbb{S}(\nu) = \mathrm{Th}(\nu) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
L & \longrightarrow & CL & \longrightarrow & \Sigma L & &
\end{array}$$

so

$$\mathrm{Th}(\nu) \simeq \Sigma L .$$

In short, the Thom space of the normal bundle is (up to suspension) the Spanier-Whitehead dual of $M_+$. This is "Milnor-Spanier duality." Atiyah [1] proved a version of this for manifolds-with-boundary and it is often called "Atiyah duality."

The duality isomorphism is thus

$$\alpha : H_{n-q}(M) \xrightarrow{\cong} \overline{H}^{q+k}(\mathrm{Th}(\nu)) .$$

Combining this with the Thom isomorphism gives an isomorphism

$$H_{n-q}(M) \xrightarrow{\cong} H^q(M) .$$

This is Poincaré duality! and indeed a proof of it can be given along these lines.

### Thom's counterexample

The duality map sends the fundamental class $[M] \in H_n(M)$ to the Thom class $U \in H^{n+k}(\mathrm{Th}(\nu))$. Thus if $\theta \in A^q$ annihilates the Thom class, we find that

$$\alpha([M]\overline{\theta}) = \theta(\alpha[M]) = \theta U = 0 ,$$

so for any $x \in H^{n-q}(M)$

$$0 = \langle x, [M]\overline{\theta} \rangle = \langle \overline{\theta} x, [M] \rangle .$$

The image of $\overline{\theta}$ in $H^n(M)$ annihilates the fundamental class.

Let $f : M \to X$ be any map, and $x \in H^{n-q}(X)$, and compute

$$\langle \overline{\theta}x, f_*[M]\rangle = \langle f^*\overline{\theta}x, [M]\rangle = \langle \overline{\theta}(f^*x), [M]\rangle = 0\,.$$

So in order for a class in $H_n(X)$ to be carried by an oriented $n$-manifold the image of $\overline{\theta}$ in $H^n(X)$ must annihilate it.

For a specific example, Thom looked at $K_1 = K(\mathbb{Z}/3\mathbb{Z}, 1)$. This is an infinite "lens space." The cohomology is

$$H^*(K_1; \mathbb{F}_3) = E[e] \otimes \mathbb{F}_3[x]\,, \quad |e| = 1\,, |x| = 2\,.$$

The Steenrod action is determined by

$$\beta e = x\,, \quad P^1 x = x^3\,.$$

The anti-automorphism is easily seen to send both $\beta$ and $P^1$ to their negatives, so

$$\overline{\beta P^1} = P^1\beta\,.$$

The class $x^3 \in H^6(K_1; \mathbb{F}_3)$ is in the image of this class, so the dual homology class cannot be carried by an oriented closed manifold.

This is mod $p$; how about integrally? The Bocksteins tell us that $\overline{H}_*(K_1; \mathbb{Z})$ is unfortunately concentrated in odd degrees, while $P^1\beta H^*(K_1; \mathbb{F}_3) = 0$ in odd degrees. So Thom moves up a dimension to $K_2 = K(\mathbb{Z}/3\mathbb{Z}, 2)$. It's known, and not hard to verify by pulling back under the map $K_1 \times K_1 \to K_2$ classifying the cup product, that $\beta P^1 \beta\iota_2 \neq 0$. In homology, then, there is a class $c \in H_8(K_2; \mathbb{F}_3)$ such that $c\beta P^1\beta \neq 0$ in $H_2(K_2; \mathbb{F}_3)$. The class $c\beta \in H_7(K_2; \mathbb{F}_3)$ can't be carried by an oriented manifold since

$$\langle P^1\beta\iota, c\beta\rangle = \langle \beta P^1(\beta\iota), c\rangle \neq 0\,.$$

But the Bockstein factors as

$$H_8(K_2; \mathbb{F}_3) \xrightarrow{\partial} H_7(K_2; \mathbb{Z}) \xrightarrow{\rho} H_7(K_2, \mathbb{F}_3)\,,$$

so $\partial c \in H_7(K_2; \mathbb{Z})$ can't be carried by a manifold since its reduction $\beta c \in H_7(K_2; \mathbb{F}_3)$ can't be.

The Postnikov system for $MSO$ provides further obstructions.

## The Brown-Peterson spectrum

The annihilator ideal of $U \in H^0(MSO)$ at an odd prime is the two-sided ideal generated by the Bockstein. The quotient by this ideal turns out to be the cohomology of a ring spectrum – not an Eilenberg Mac Lane spectrum, but rather a new gadget called the "Brown-Peterson spectrum" and denoted (without reference to the prime $p$) by $BP$. (Frank Peterson, 1930–2000, was an MIT faculty member and long-time treasurer of the AMS.) At odd primes, $MSO$ splits into a product of suspensions of $BP$. The mod $p$ Thom class restricts to a map $BP \to H\mathbb{F}_p$ that induces an embedding of $H_*(BP) \hookrightarrow A_*$ as the polynomial algebra on the $\xi$'s.

The homotopy type of $MU$ was studied by Milnor using the Adams spectral sequence. It turns out that

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \ldots]\,, \quad |x_i| = 2i\,.$$

It turns out that $MU$ localized at any prime $p$ splits as a product of the $p$-local Brown-Peterson spectrum as well (even if $p = 2$). The homotopy of $BP$ is also a polynomial algebra, but now much sparser:

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]\,, \quad |v_i| = 2p^i - 2\,.$$

## Surgery

There is a simple way to modify a manifold to give a new manifold with different topology but related by a cobordism. The most classical example of surgery occurs in dimension 2. Start with an embedded loop $L$ in a closed surface $M$. Assume that the normal bundle of $L$ is framed (always the case if $M$ is orientable), so that we have an embedding of $S^1 \times D^1$ into $M$. This kind of product is familiar! In general

$$\partial(D^p \times D^q) = S^{p-1} \times D^q \cup_{S^{p-1}\times S^{q-1}} D^p \times S^{q-1} .$$

In our case $p = 2$ and $q = 1$. We can remove the interior of $\partial D^2 \times D^1$ and replace it with the interior of $D^2 \times \partial D^1 = D^2 \times S^0$, to get a new manifold $M'$. If the regular neighborhood of the loop was a belt around a waste (or "handle"), this has the effect of removing the belt and capping off the two body parts. This process is called "surgery."

What's a little harder to see is that $D^p \times D^q$ can be used to construct a cobordism between $M$ and $M'$.

A proof using Morse theory [26] shows that any two closed $n$ manifolds in the same bordism class can be connected by a bordism constructed by a series of surgeries.

The surgery operation, pioneered by Milnor and Wallace and later Browder, Novikov, and Wall, led to an enormous research program aimed at the classification of manifolds up to diffeomorphism.

**Exercise 40.2.** Show that any positive dimensional oriented bordism class contains a connected manifold. Show that any oriented cobordism class of dimension at least 2 contains a simply connected manifold. Display counterexamples to these to statements in lower dimensions.

**Remark 40.3.** The surgery process involves killing homology groups in a manifold. It requires establishing that (1) the class is spherical – in the image of the Hurewicz map; (2) the map from a sphere is a smooth embedding; and (3) the normal bundle of this embedded sphere is trivial.

Typically the first requirement is met using the Hurewicz theorem; we try to kill bottom dimensional homology. The second can be achieved by Whitney embedding theorem as long as we are below the middle dimension of the manifold. The third is much more problematic. One way to ensure that the process can continue above dimension one is to work with framed bordism. The Pontryagin-Thom theorem identifies this with stable homotopy, so there is considerable interest in this case. The surgery process then works to find a "highly connected" representative of a framed bordism class in which the homology is concentrated in the middle dimension. When $n$ is odd, any class in $\Omega_n^{fr}$ has is represented by a homotopy sphere, since there is then no middle dimension. The same turns out to be true when $n = 4k$. When $n = 4k + 2$, there is a potential obstruction, the *Kervaire invariant*, with values in $C_2$. It's already visible in dimension 2, when the square of the nontrivially framed circle (which represents the stable homotopy class $\eta$ of the Hopf map $S^3 \to S^2$) is not framed null-bordant (since in fact $\eta^2 \neq 0$). The higher dimensional Hopf fibrations give other examples in dimensions 6 and 14. William Browder proved that the invariant could be nonzero only in dimensions of the form $2^j - 2$, and identified the invariant in terms of the Adams spectral sequence. In the 1970's examples were constructed using homotopy theory in dimensions 30 and 62, and in 2015 work of Mike Hill, Mike Hopkins, and Doug Ravenel finally showed that the invariant is in fact trivial for dimensions larger than 126 (where it remains unknown today).

## Signature

This ability to move around within a cobordism class suggests that there are very few bordism invariants that one an derive from cohomology. What homological features of a manifold are cobordism invariants?

When $M$ is an oriented $4k$-manifold, $H^{2k}(M;\mathbb{Q})$ supports a symmetric bilinear form, the "intersection form"

$$x \cdot y = \langle xy, [M] \rangle$$

which is nondegenerate on account of Poincaré duality. A fact from linear algebra: Any symmetric bilinear form over $\mathbb{Q}$ is diagonal with respect to some basis. If it is nondegenerate then all the diagonal entries in the diagonalization are nonzero, and the difference between the number of positive entries and the number of negative entries is a independent of the diagonalizing basis. It is the *signature* of the bilinear form.

**Lemma 40.4** (Thom)**.** *The signature of the intersection form of an oriented $4k$-manifold is a multiplicative oriented bordism invariant.*

This follows from Lefschetz duality and the Künneth theorem. The result is a graded ring homomorphism

$$\sigma : \Omega_*^{SO} \to \mathbb{Z}[u] \,, \quad |u| = 4 \,.$$

Such a ring homomorphism is a *genus*. (This term entered mathematics from biology through Gauss's work on quadratic forms, and then spread to the genus of a surface, and then to other numerical invariants of manifolds.) Since the characteristic number map is a rational isomorphism, the value of a rational genus on a $4k$-manifold $M$ is some Pontryagin number.

Since the even complex projective spaces generate $\Omega_*^{SO}$ rationally, giving the value of a genus on them completely specifies the value of the genus on any oriented manifold. Since $\mathbf{C}P^{2k}$ obviously has signature 1 for any $k$, the signature is in a sense the simplest genus. For each $k$ there is a polynomial

$$L_k \in H^{4k}(BSO;\mathbb{Q})$$

in the Pontryagin classes such that for any closed oriented $4k$-manifold $M$

$$\sigma(M) = \langle L_k(\nu_M), [M] \rangle \,.$$

This is the "Hirzebruch signature theorem." Identifying these polynomials is a beautiful story. The results are for example that

$$L_1 = \frac{p_1}{3} \,, \quad L_2 = \frac{7p_2 - p_1^2}{45} \,, \quad L_3 = \frac{62p_3 - 13p_2p_1 + p_1^3}{945} \,, \ldots \,.$$

These formulas put divisibility conditions on certain combinations of Pontryagin classes of the normal bundle of an embedding of a closed manifold into Euclidean space: while the $L$-class has denominators, you get an integral class when you pair it against the fundamental class. The first normal Pontryagin class of an orientable 4-manifold has to be divisible by 3, for example.

The signature theorem in dimension 8 played a key role in Milnor's proof that certain $S^3$-bundles over $S^4$ are not diffeomorphic to the standard 7-sphere despite being homeomorphic to it.

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