18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Generating Analytic Functors (Lecture 10)

Let $\mathrm{Fun}^{\mathrm{an}}$ denote the category of analytic functors from Vect^f to Vect, as defined in the last lecture. Our goal in this lecture is to prove the following, which we stated without proof in the last lecture:

Theorem 1. *The abelian category* $\mathrm{Fun}^{\mathrm{an}}$ *is generated by the divided power functors* $\{\Gamma^n\}_{n\geq 0}$.

Let us say that a functor $F : \mathrm{Vect}^f \to \mathrm{Vect}$ is *good* if there exists a surjection

$$\oplus_\alpha \Gamma^{n_\alpha} \to F.$$

Clearly, every good functor is analytic. Theorem 1 asserts the converse: every analytic functor is good. We observe that the collection of good functors is stable under quotients and direct sums. Consequently, every colimit of good functors is good. Since every analytic functor F is the direct limit of its polynomial subfunctors $F^{(n)} \subseteq F$, Theorem 1 can be reformulated as follows:

Proposition 2. *Every polynomial functor* $F : \mathrm{Vect}^f \to \mathrm{Vect}$ *is good.*

Now recall the projective objects $P_V \in \mathrm{Fun}$, defined in the last lecture by the formula $P_V(W) = \mathbf{F}_2\{\mathrm{Hom}_{\mathbf{F}_2}(V, W)\}$. These functors are not analytic, but they generate the category Fun of *all* functors from Vect^f to Vect. In particular, we have a surjection

$$\oplus_{i \in I} P_{V_i} \to F$$

in the category Fun. In particular, we can write F as a filtered colimit of subfunctors

$$F_{I_0} = \mathrm{i}m(\oplus_{i\in I_0} P_{V_i} \to F) \subseteq F.$$

where I_0 ranges over finite subsets of I. Since the collection of good functors is stable under colimits, it will suffice to show that each F_{I_0} is a good functor. Proposition 2 is now an immediate consequence of the following assertion:

Proposition 3. *Let F be a polynomial functor, and suppose there exists a surjection*

$$\oplus_{i\in I} P_{V_i} \to F,$$

where the set I is finite. Then there exists a surjection

$$\oplus_{\alpha \in A} \Gamma^{n_\alpha} \to F,$$

where the set A is finite.

Let us say that a functor $G \in \mathrm{Fun}$ is *locally finite* if $G(V)$ is finite dimensional for each $V \in \mathrm{Vect}^f$. The functors P_{V_i} are locally finite, as are the functors $\{\Gamma^n\}_{n\geq 0}$. The duality functor D induces a (contravariant) equivalence from the category of locally finite functors to itself. Moreover, we observe that if G is locally finite, then we have an equality of dimension functors $d_G = d_{DG}$, so that G is polynomial if and only if G is polynomial. Proposition 3 can therefore be reformulated in the following dual form:

Proposition 4. *Let F be a polynomial functor, and suppose that there exists an injection*

$$F \hookrightarrow \oplus_{i \in I} I_{V_i}$$

where I is finite. Then there exists an injection

$$F \hookrightarrow \oplus_{\alpha \in A} \operatorname{Sym}^{n_\alpha}$$

where the set A is finite.

For each $i \in I$, let F_i denote the image of F in I_{V_i}. Then F_i is a quotient of F, and therefore a polynomial functor. Moreover, we have an inclusion $F \hookrightarrow \oplus_{i \in I} F_i$. It will therefore suffice to prove Proposition 4 after replacing F by each F_i; in other words, we may suppose that I consists of a single element. We are therefore reduced to proving the following special case of Proposition 4:

Proposition 5. *Let V be a finite dimensional vector space over $\mathbf{F}_2$, and let F be a polynomial subfunctor of I_V. Then there exists an injection*

$$F \hookrightarrow \oplus_{\alpha \in A} \operatorname{Sym}^{n_\alpha}$$

where the set A is finite.

Let us now suppose that V has dimension n, so that the functor I_V can be written as a tensor product

$$I_{\mathbf{F}_2} \otimes I_{\mathbf{F}_2} \otimes \ldots \otimes I_{\mathbf{F}_2}.$$

As we observed last time, there is a canonical surjection

$$\operatorname{Sym}^* \simeq \oplus_{n \geq 0} \operatorname{Sym}^n \to I_{\mathbf{F}_2}.$$

For $0 < n \leq \infty$, let $S^n \subseteq I_{\mathbf{F}_2}$ denote the image of the direct sum

$$\oplus_{1 \leq i \leq n} \operatorname{Sym}^i$$

in $I_{\mathbf{F}_2}$. Then we have a direct sum decomposition $I_{\mathbf{F}_2} \simeq \operatorname{Sym}^0 \oplus S^\infty$, so that I_V can be written as a finite sum $\oplus_{j \in J} (S^\infty)^{\otimes n_j}$. We now apply our previous argument: for each $j \in J$, let F_j denote the image of F in $(S^\infty)^{\otimes n_j}$. Then we have a monomorphism $F \to \oplus_{j \in J} F_j$, and it will suffice to prove the result after applying F by F_j. In other words, we may reformulate Proposition 5 as follows:

Proposition 6. *Let F be a polynomial functor, and suppose there exists an injection*

$$F \hookrightarrow S^\infty \otimes \ldots \otimes S^\infty.$$

Then there exists an injection

$$F \hookrightarrow \operatorname{Sym}^m$$

for some $m \geq 0$.

We observe that the functor S^∞ is the direct limit of the subfunctors $\{S^k \subseteq S^\infty\}_{k \geq 1}$. Consequently, F is the direct limit of the subfunctors

$$F \cap (S^k \otimes S^k \otimes \ldots \otimes S^k).$$

We claim that one of these subfunctors must coincide with F. This is a consequence of the following general fact:

Lemma 7. *Every locally finite polynomial functor F is a Noetherian object of* Fun*: in other words, there are no infinite ascending chains of subfunctors*

$$F_0 \subset F_1 \subset \ldots \subseteq F.$$

Proof. Let F be a polynomial functor of degree $\leq m$. Then the dimension function d_F is a polynomial of degree $\leq m$, and is therefore determined by its values on $\{0, 1, \ldots, m\}$. If we have an inclusion $F' \subseteq F$, then F' is also a polynomial of degree $\leq m$ and we have an inequality

$$d_{F'}(0) + \ldots + d_{F'}(m) \leq d_F(0) + \ldots + d_F(m)$$

If equality holds, then $d_{F'} = d_F$, so that $F' = F$. Thus every chain of proper subfunctors of F has length at most $d_F(0) + \ldots + d_F(m)$. □

We can now Proposition 6 to the following:

Proposition 8. *Let F be a functor of the form $(S^k)^{\otimes n}$, where $k \geq 1$ and $n \geq 0$. Then there exists a monomorphism $F \hookrightarrow \mathrm{Sym}^m$ for some $m \geq 0$.*

Proposition 8 is obvious if $n = 0$ (in that case, $F \simeq \mathrm{Sym}^0$ and we can take $m = 0$). The main difficulty is in the case $n = 1$, where we need the following result:

Proposition 9. *Let $k \geq 0$. Then there exists a monomorphism of functors $S^{k+1} \hookrightarrow \mathrm{Sym}^{2^k}$.*

We can reduce the general case to Proposition 9 using the following lemma:

Lemma 10. *Let $m, m' > 0$. Then there exists $m'' > 0$ and a monomorphism of functors*

$$\mathrm{Sym}^m \otimes \mathrm{Sym}^{m'} \to \mathrm{Sym}^{m''}.$$

Proof. For $q \geq 0$, we have an iterated Frobenius map

$$\mathrm{Sym}^m \to \mathrm{Sym}^{2^q m}$$

$$f \mapsto f^{2^q}.$$

It now suffices to observe that the composite map

$$\mathrm{Sym}^m \otimes \mathrm{Sym}^{m'} \to \mathrm{Sym}^{2^q m} \otimes \mathrm{Sym}^{m'} \hookrightarrow \mathrm{Sym}^{w^q m + m'}$$

is a monomorphism for $2^q > m'$. □

We now prove Proposition 9 using an explicit construction of Kuhn. We note that the kernel of the map $\mathrm{Sym}^*(V) \to I_{\mathbf{F}_2}(V)$ is generated by $v^2 - v$ for $v \in V$. We can therefore describe the space $S^{k+1}(V)$ as the quotient of $\oplus_{1 \leq d \leq k+1} V^{\otimes d}$ by the following relations:

(1) If $\sigma \in \Sigma_d$ is a permutation, then

$$v_1 \otimes \ldots \otimes v_d = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

in $S^{k+1}(V)$.

(2) If $d < k$, and $v \in V$, then

$$v_1 \otimes \ldots \otimes v_d \otimes v = v_1 \otimes \ldots \otimes v_d \otimes v \otimes v$$

in $S^{k+1}(V)$.

We define a map $\theta : \oplus_{1 \leq \leq k+1} V^{\otimes d} \to \mathrm{Sym}^{2^k}(V)$ by the formula

$$\theta(v_1 \otimes \ldots \otimes v_d) = \sum_{2^{i_1} + \ldots + 2^{i_d} = 2^k} v_1^{2^{i_1}} \ldots v_d^{2^{i_d}},$$

It is clear that θ is compatible with the relations of type (1). We claim that θ is also compatible with relations of the type (2): that is, if $d < k$, then

$$\theta(v_1 \otimes \ldots \otimes v_d \otimes v) = \theta(v_1 \otimes \ldots \otimes v_d \otimes v \otimes v).$$

To prove this, we observe that the right hand side is a sum of terms associated to sequences of integers $(i_1, \ldots, i_d, j, k)$ where $2^k = 2^{i_1} + \ldots + 2^{i_d} + 2^j + 2^k$. The terms associated to $(i_1, \ldots, i_d, j, k)$ and $(i_1, \ldots, i_d, k, j)$ cancel if $j \neq k$, while the terms associated to the sequence $(i_1, \ldots, i_d, j, j)$ appear on the left hand side as associated to the sequence $(i_1, \ldots, i_d, j+1)$. To complete the proof, it will suffice to show that no other terms appear on the left hand side. In other words, we must show that if $2^{i_1} + \ldots + 2^{i_d} + 2^j = 2^k$, then $j > 0$. If not, we have $2^{i_1} + \ldots + 2^{i_d} = 2^k - 1$, which has k nonzero digits in its base 2-expansion. Since each term in the sum is a power of 2, the sum must include at least k terms, which contradicts our assumption that $d < k$.

We have now shown that θ is compatible with the relations (1) and (2), and therefore defines a map $\overline{\theta} : S^{k+1}(V) \to \mathrm{Sym}^{2^k}(V)$. This map is evidently functorial in V, and so defines a natural transformation of functors $\psi : S^{k+1} \to \mathrm{Sym}^{2^k}$. To complete the proof of Proposition 9, it will suffice to show that this natural transformation is a monomorphism. In other words, we must show that each of the maps $S^{k+1}(V) \to \mathrm{Sym}^{2^k}(V)$ is injective.

To prove this, we will need the following lemma:

Lemma 11. *Let F and F' be nonzero subfunctors of S^∞. Then $F \cap F' \neq 0$.*

Proof. We compute that the endomorphism ring

$$R = \mathrm{Hom}_{\mathrm{Fun}}(I_{\mathbf{F}_2}, I_{\mathbf{F}_2}) \simeq I_{\mathbf{F}_2}(\mathbf{F}_2)^\vee \simeq \mathbf{F}_2 \oplus \mathbf{F}_2$$

has dimension 2 over the field $\mathbf{F}_2$. The endomorphism ring S^∞ is properly contained in R, and therefore has dimension 1 over $\mathbf{F}_2$. It follows that every nonzero endomorphism of S^∞ is an isomorphism.

Suppose $F \cap F' = 0$. Then the induced map $F \to S^\infty/F'$ is a monomorphism. Since S^∞ is injective, we can solve the lifting problem depicted in the diagram

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, hook] & S^\infty \\ S^\infty/F'. \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Composing f with the projection map $S^\infty \to S^\infty/F'$, we obtain an endomorphism $\overline{f} : S^\infty \to S^\infty$. This endomorphism is not an isomorphism, since $\overline{f}|F' = 0$. Consequently, $\overline{f} = 0$. Since $\overline{f}|F$ is the identity, we deduce that $F = 0$, a contradiction. $\square$

Let us apply Lemma 11 in the case $F = \mathrm{Sym}^1 \simeq S^1 \subseteq S^\infty$ and $F' = \ker(\psi)$. If ψ is not a monomorphism, then $\ker(\psi)$ is nonzero, so $F \cap F' \neq 0$. In other words, for some vector space V the composite map

$$\mathrm{Sym}^1(V) \to S^{k+1}(V) \xrightarrow{\overline{\theta}} \mathrm{Sym}^{2^k}(V)$$

is not injective. But this map is simply the iterated Frobenius $v \mapsto v^{2^k}$, and we obtain a contradiction.