SYMPLECTIC GEOMETRY, LECTURE 10

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1. Curvature and the Covariant Derivative

Let ∇ be a connection, $R^\nabla \in \Omega^2(M, \text{End } E)$ its curvature, where

$$R^\nabla(u,v)s = \nabla_u\nabla_v s - \nabla_v\nabla_u s - \nabla_{[u,v]}s \tag{1}$$

Last time, we saw that in a local trivialization, $\nabla = d + A$, where A is a 1-form with values in $\text{End}(E)$, and $R^\nabla = dA + A \wedge A$. Moreover, a change of basis given by $g \in C^\infty(U, \text{End}(E))$ acts by

$$A \mapsto g^{-1}Ag + g^{-1}dg, R^\nabla \mapsto g^{-1}R^\nabla g \tag{2}$$

We can extend the covariant derivative $\nabla : C^\infty(M,E) \to \Omega^1(M,E)$ to an operator $d^\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$. Locally, $\Omega^p(M,E)$ is given by sums $\sum \alpha_i s_i$, where $\alpha_i = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ are p-forms and $e_i = s_{i_1\cdots i_p}$ are sections of E, and d^∇ maps this to $\sum(\nabla s_i) \wedge \alpha_i + s_i d\alpha_i$. In a trivialization $\nabla = d + A$, we have

$$d^\nabla \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = d \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + A \wedge \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} \tag{3}$$

That is, $d^\nabla = d + A \wedge (\cdot)$.

Proposition 1. *$R^\nabla = (d^\nabla)^2 : \Omega^0(M,E) \to^{d^\nabla} \Omega^1(M,E) \to^{d^\nabla} \Omega^2(M,E)$. More generally,*

$$R^\nabla \wedge \cdot = (d^\nabla)^2 : \Omega^p(M,E) \to^{d^\nabla} \Omega^{p+1}(M,E) \to^{d^\nabla} \Omega^{p+2}(M,E) \tag{4}$$

Proof. In a local trivialization,

$$\begin{aligned} d^\nabla(d^\nabla\alpha) &= d^\nabla(d\alpha + A\wedge\alpha) = d(d\alpha + A \wedge \alpha) + A \wedge (d\alpha + A \wedge \alpha) \\ &= (dA) \wedge \alpha - A \wedge d\alpha + A \wedge d\alpha + A \wedge A \wedge \alpha = (dA + A \wedge A) \wedge \alpha \end{aligned} \tag{5}$$

as desired. □

Remark. R^∇ can be thought of as an obstruction for $0 \to C^\infty(E) \overset{d^\nabla}{\to} \Omega^1(E) \overset{d^\nabla}{\to} \cdots$ being a complex. If the manifold is flat, i.e. $R^\nabla = 0$, then we obtain a twisted de Rham cohomology with coefficients in E. R^∇ is also an obstruction to the integrability of the horizontal distribution $\mathcal{H}^\nabla$, i.e. homotopy invariance of parallel transport.

When $E = TM$ for (M,g) a Riemannian manifold, there is a unique metric ($X \cdot g(u,v) = g(\nabla_X u, v) + g(u, \nabla_X v)$) connection on TM s.t. $\nabla_X Y - \nabla_Y X = [X,Y]$, called the *Levi-Cevita* connection. Now, let (M, ω, g, J) be a symplectic manifold with a compatible almost complex structure. Then TM is a complex vector bundle, but ∇^{LC} is not $\mathbb{C}$-linear in general. Indeed, it is $\mathbb{C}$-linear $\Leftrightarrow \nabla J = 0$ for the induced connection ∇ on $\text{End}(TM) \Leftrightarrow J$ is integrable (i.e. an actual complex structure).

2. Complex Vector Bundles and Chern Classes

Let $L \to M$ be a complex line bundle, ∇ a connection (possibly Hermitian w.r.t. a Hermitian metric $\langle\cdot,\cdot\rangle$). In a local trivialization, $R^\nabla = dA \in \Omega^2(M,\mathbb{C})$ (resp. $\Omega^2(M, i\mathbb{R})$) since $A \in \Omega^1(U,\mathbb{C})$ (resp. $\Omega^1(M, i\mathbb{R})$) has $A \wedge A = 0$. Thus, R^∇ is a closed 2-form, and has a corresponding class $c = [R^\nabla] \in H^2(M,\mathbb{C})$ (resp. $\Omega^2(M, i\mathbb{R})$). For ∇' another connection, we have a global decomposition $\nabla' = \nabla + a$ for $a \in \Omega^1(M,\mathbb{C})$, so $R^{\nabla'} = R^\nabla + da$ and $[R^\nabla] = [R^{\nabla'}]$. Thus, c is an invariant of L independent of ∇ in $H^2(M,\mathbb{C})$ (resp. $H^2(M, i\mathbb{R})$). Since we can always choose a connection compatible with a given Hermitian form, we have

Definition 1. *The* first Chern class *of* L *is* $c_1(L) = [\frac{1}{2\pi} R^\nabla] \in H^2(M, \mathbb{R})$.

Remark. From algebraic topology, we can obtain an associated integer class $c_1(L) \in H^2(M, \mathbb{Z})$ corresponding to this form.

Now, let $E \to M$ be a complex vector bundle with connection ∇.

Definition 2. *The* total Chern form *is*

$$c(E, \nabla) = \det\ (I + \frac{i}{2\pi} R^\nabla) \in \bigoplus_{p\ even} \Omega^p(M, \mathbb{C}) \tag{6}$$

Decomposing this element, we obtain projections $c_j(E, \nabla) \in \Omega^{2j}(M, \mathbb{C})$. Here $I + \frac{i}{2\pi} R^\nabla$ is a matrix with entries (const + 2-forms) in a local trivialization, and det is the usual determinant under the $\wedge$ product. As before, this is independent of change of basis.

Remark. By the formula for $\det\ (I + tM) = 1 + t \cdot \mathrm{Tr}(M) + \cdots$, we find that $c_1(E, \nabla) = \frac{i}{2\pi} \mathrm{Tr}(R^\nabla)$, and

$$c_r(E, \nabla) = \left(\frac{i}{2\pi}\right)^r \det\ R^\nabla \tag{7}$$

We can do the same for any ad-invariant polynomial in R^∇, giving Chern-Weil theory (for complex vector bundles, simply get functions of $c_1, \ldots, c_r$).

Theorem 1. $c_j(E, \nabla)$ *is closed, and* $c_j(E) = [c_j(E, \nabla)] \in H^{2j}(M, \mathbb{R})$ *is independent of* ∇.

Proof. Closedness follows from the Bianchi identity for $d^\nabla(R^\nabla)$, and independence follows from showing that $c_j(E, \nabla') - c_j(E, \nabla)$ is a sum of exact terms. □

Remark. Another approach involves the *Euler class* of an oriented rank k real vector bundle $E \to M$ over a compact, oriented manifold M. Let s be a section of E, chosen so s is transverse to the zero section and $Z = s^{-1}(0)$ is a smooth, oriented submanifold of codimension k. Then, at a point of Z, $\nabla s : NZ \to E|_Z$ is an isomorphism. We define $e(E) = [Z] \in H_{n-k}(M, \mathbb{Z}) \cong H^k(M, \mathbb{Z})$ by Poincaré duality. If E was a rank r $\mathbb{C}$-vector bundle, then $c_r(E) = e(E)$.

Remark. For $TM \to M, e(TM) \in H^n(M, \mathbb{Z}) = \mathbb{Z} \Leftrightarrow \chi(M) = e(TM) \cdot [M]$. Moreover, for E, ∇ a flat connection, $c_j(E) = 0 \in H^{2j}(M, \mathbb{R})$.