SYMPLECTIC GEOMETRY, LECTURE 19

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We now return to the complex Kähler case. Let (M, ω, J) be a complex Kähler manifold.

Proposition 1 (Donaldson). *$\exists$ a family of sections $(\sigma_{k,p})_{k>k_0, p\in M}$ which is uniformly bounded and almost-holomorphic, uniformly concentrated, and satisfies $|\sigma_{k,p}| \geq c > 0$ on $B(p, k^{-1/2})$. Furthermore, $\exists$ a family of holomorphic sections $(\tilde{\sigma}_{k,p})$ with $\sup|\sigma_{k,p} - \tilde{\sigma}_{k,p}|, \sup(k^{1/2}|\nabla\sigma_{k,p} - \nabla\tilde{\sigma}_{k,p}|) \leq O(\exp(-\lambda k^{1/3}))$. That is, the $\tilde{\sigma}_{k,p}$ are so close to $\sigma_{k,p}$ that they're interchangeable in practice.*

Proof. Fix $p \in M$ and holomorphic coordinates $(M, p) \to (\mathbb{C}^n, 0)$ (not necessarily Darboux). We can choose the coordinates to be isometric at the origin.

(1) Let u be a local section of L near p which is holomorphic and s.t. $|u(x)| = 1$ (e.g. $u \equiv 1$ in a holomorphic trivialization). Then

$$\overline{\partial}\partial \log|u|^2 = \overline{\partial}(u^{-1}\partial^\nabla u) = u^{-1}\overline{\partial}^\nabla\partial^\nabla u = R^{1,1} = -i\omega \tag{1}$$

with the third equality coming from $(R^\nabla)^{1,1} = \overline{\partial}^\nabla\partial^\nabla + \partial^\nabla\overline{\partial}^\nabla = (R^\nabla)^{1,1}$ and $\overline{\partial}^\nabla u = 0$. In local coordinates, we can write

$$\log|u|^2 = \sum_j (f_j z_j + \overline{f}_j \overline{z}_j) + \sum_{ij}(g_{ij} z_i \overline{z}_j + h_{ij} z_i z_j + \overline{h}_{jk}\overline{z}_i\overline{z}_j) + O(|z|^3) \tag{2}$$

Replacing u by $\exp(\sum -f_j z_j - \sum h_{ij} z_i z_j)u$ (which preserves holomorphicity), we can assume $\log|u|^2 = \sum g_{ij} z_i \overline{z_j} + O(|z|^3)$. $\overline{\partial}\partial\log|u|^2 = -i\omega \implies (g_{ij}) = -\frac{1}{2}(\text{metric tensor on } T_xM) \implies \log|u|^2 = -\frac{1}{2}|z|^2 + O(|z|^3)$. Hence u^k is a local holomorphic section of $L^{\otimes k}$, $|u^k| = \exp(-\frac{k}{4}|z|^2 + kO(|z|^3))$. Estimating the growth of derivatives of $\log|u|^2$ gives us uniform concentratedness estimates as long as $|z| << 1$ (which is fine since the "support" of $u^k \sim$ a ball of radius $\frac{1}{\sqrt{k}}$). Then let $\sigma_{k,p}(q) = \chi_k(\text{dist}(p,q))u(q)^k$, where χ_k is a smooth cut-off function at distance $\sim k^{-1/3}$ (i.e. $\chi_k \equiv 1$ inside the ball of radius $k^{-1/3}$ and 0 outside a larger ball).

Note that the cutoff occurs in the region where $|z| \sim k^{-1/3}$ i.e. $|u^k| \sim \exp(-k\frac{|z|^2}{4}) \sim \exp(-k^{1/3})$. Thus we get $\sup|\overline{\partial}\sigma_{k,p}| = \sup|u^k\overline{\partial}(\chi_k)| \leq O(\exp(-\lambda k^{1/3}))$ since $\overline{\partial}\chi_k \equiv 0$ except for $|z| \sim k^{-1/3}$ and $|\overline{\partial}\chi_k| \leq k^{1/3}$.

(2) To obtain the $\tilde{\sigma}_{k,p}$, we use the following lemma:

Lemma 1. *$\forall s \in \Gamma(L^{\otimes k}), \exists \xi \in \Gamma(L^{\otimes k})$ s.t. $||\xi||_{L^2} \leq \frac{c}{\sqrt{k}}||\overline{\partial}s||_{L^2}$ and $s + \xi$ is holomorphic.*

We apply this lemma to $\sigma_{k,p}$ and obtain $||\xi||_{L^2} \leq \frac{c}{\sqrt{k}}||\overline{\partial}\sigma_{k,p}||_{L^2} \leq O(k^{-2n/3-1/2}\exp(-\lambda k^{-1/3}))$, where the L^2 estimate on $\overline{\partial}\sigma_{k,p}$ follows from the pointwise bound and the observation that it is supported in a ball of volume $\sim k^{-2n/3}$. To get a pointwise C^r-estimate on ξ, we use a Cauchy estimate expressing values of holomorphic functions at q by integrals over balls containing q. At points inside $B(p, k^{-1/3})$, $\chi = 1$ so $\sigma_{k,p}$ is holomorphic there, as is ξ, and $||\xi||_{C^r}$ is controlled by $||\xi||_{L^2} \sim \exp(-\lambda k^{1/3})$ on $B(k^{-1/3})$. Finally, the Cauchy estimates for $\sigma_{k,p} + \xi$ imply that $||\sigma_{k,p} + \xi||_{C^r}$ is also controlled by the local L^2 norm and thus also bounded by $\exp(-\lambda k^{1/3})$ outside of $B(p, k^{-1/3})$ as desired.

□

Proof of Lemma. We use the operator $\Delta_k = \overline{\partial}^*_{L^k}\overline{\partial}_{L^k} + \overline{\partial}_{L^k}\overline{\partial}^*_{L^k} : \Omega^{0,1}(L^{\otimes k}) \to \Omega^{0,1}(L^{\otimes k})$. We estimate via a Weitzenböck formula: fixing a tangent frame e_i of $T^{1,0}$, e^i the dual frame, we have

$$\begin{aligned}\overline{\partial}\alpha &= \sum_i \overline{e^i} \wedge \nabla_{\overline{e_i}}\alpha \\ \overline{\partial}^*\alpha &= -\sum_i g(e^i, \overline{e^j}) i_{\overline{e_j}}(\nabla_{e_i}\alpha)\end{aligned} \tag{3}$$

Take a frame that's orthonormal at the origin, and radially parallel transport so $\nabla_{e_i} e_j = 0$ at the origin; this preserves type $(1,0)$ forms since J is integrable. Then

$$\begin{aligned}\Delta_k\alpha &= -\sum_{ij} i_{\overline{e_i}}(\overline{e_j} \wedge \nabla_{e_i}\nabla_{\overline{e_j}}\alpha) - \sum_{ij} \overline{e^j} \wedge (i_{\overline{e_i}}\nabla_{\overline{e_j}}\nabla_{e_i}\alpha) \\ &= \sum_i -\nabla_{e_i}\nabla_{\overline{e_i}}\alpha + \sum_{ij} \overline{e^j} \wedge i_{\overline{e_i}}(R^{T^*M\otimes L^k}(e_i, \overline{e_k})\alpha) \\ &= D\alpha + R\alpha + k\alpha\end{aligned} \tag{4}$$

because at the origin $R^{L^k}(e_i, \overline{e_j}) = -ik\omega(e_i, \overline{e_j}) = k\delta_{ij}$. D is semipositive, since $\langle D\alpha, \alpha\rangle = \sum ||\nabla_{\overline{e_i}}\alpha||^2 + d(\text{something}) \implies \int_M \langle D\alpha, \alpha\rangle \geq 0$. Therefore, for k large enough, Δ_k is invertible and $\exists$ an inverse G of norm $O(\frac{1}{k})$.

Given $s \in \Gamma(L^k)$, set $\xi = -\overline{\partial}^* G\overline{\partial}s$. Then

(1) $(s+\xi)$ is holomorphic since

$$\overline{\partial}(s+\xi) = \overline{\partial}s - \overline{\partial}\overline{\partial}^*G\overline{\partial}s = \overline{\partial}s - (\Delta_k - \overline{\partial}^*\overline{\partial})G\overline{\partial}s = \overline{\partial}^*\overline{\partial}G\overline{\partial}s \tag{5}$$

but $\mathrm{Im}\overline{\partial} \cap \mathrm{Im}\overline{\partial}^* = 0$ by Hodge theory, so $\overline{\partial}(s+\xi) = 0$.

(2) $||\xi||^2_{L^2} = \langle \overline{\partial}^*G\overline{\partial}s, \overline{\partial}^*G\overline{\partial}s\rangle = \langle \overline{\partial}\overline{\partial}^*G\overline{\partial}s, G\overline{\partial}s\rangle = \langle \overline{\partial}s, G\overline{\partial}s\rangle \leq ||G||\,||\overline{\partial}s||^2_{L^2} \leq ck^{-1}\,||\overline{\partial}s||^2_{L^2}$.

This completes the proof. □

Going from these collections of sections to the Kodaira embedding is straightforward:

- Well-definedness: we need that $\forall p, \exists s \in H^0(L^k)$ s.t. $s(p) \neq 0$, which comes from the fact that $|\tilde{\sigma}_{k,p}(p)| \simeq 1 \neq 0$.
- Immersion: need that $\forall p \in M, v \in T_pM$, $\exists \sigma_1, \sigma_2 \in H^0(L^k)$ s.t. $d_v(\frac{\sigma_1}{\sigma_2}) \neq 0$. This would give us a projection to a certain $\mathbb{CP}^1$ factor of $\mathbb{CP}^n$ which has nonzero derivative in the direction of v. We could do this by looking at $\tilde{\sigma}_{k,q_\pm}, q_\pm = \exp_p(\pm k^{-1/2}v)$. More simply, we set $\sigma_2 = \tilde{\sigma}_{k,p}, \sigma_1$ obtained by a similar process starting from $z_1\sigma_{k,p}$ (rotating the coordinates so v is along the z_1-axis) and adding ξ perturbation to make it holomorphic. Then $\frac{\sigma_1}{\sigma_2} = z_1 + \cdots \implies d_v(\frac{\sigma_1}{\sigma_2}) \neq 0$.
- Injectivity: If p, q are at a distance $<< k^{-1/3}$ then (using the above argument for immersiveness) the sections are different at p and q. If the distance is greater, $[\tilde{\sigma}_{k,p} : \tilde{\sigma}_{k,p}] \sim [1:0]$ and $[0:1]$ respectively.